Massachusetts Institute of Technology
6.042J/18.062J, Fall '02: Mathematics for Computer Science
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Solutions to In-Class Problems — Week 1, Fri

Problem 1. Translate the parts of the following deductive arguments into propositional logic notation using logical operators:

$$\begin{aligned} \wedge &::= \text{AND}, \\ \vee &::= \text{OR}, \\ \neg &::= \text{NOT}, \\ \longrightarrow &::= \text{IMPLIES}, \\ \longleftrightarrow &::= \text{IFF (if and only if)} \end{aligned}$$

This may require that you "pin down" a statement that could be interpreted in more than one way.

Identify the antecedents and conclusions of the arguments, and determine which are sound deductions and which are not. If the deduction is unsound, demonstrate a possible scenario in which all the antecedents hold but the conclusion does not.

Note: There are several inequivalent, reasonable ways to interpret several of these statements.

(a) The main course will be beef or fish. The vegetable will be peas or corn. We will not have both fish as a main course and corn as a vegetable. Therefore, we will not have both beef as a main course and peas as a vegetable.

Solution. The deduction is:

$$\frac{B \vee F, \quad C \vee P, \quad \neg(F \wedge C)}{\neg(B \wedge P)}$$

where

$$\begin{aligned} B &::= \text{"The main course will be beef."} \\ F &::= \text{"The main course will be fish."} \\ C &::= \text{"The vegetable will be corn."} \\ P &::= \text{"The vegetable will be peas."} \end{aligned}$$

This deduction is invalid. For example, $B \wedge \neg F \wedge C \wedge P$ is consistent with the antecedents but not with the conclusion. Note that as formalized, there need not be only one main course and only one vegetable; it is possible, for example, for the vegetable to be both corn and peas, as in the scenario given.

If we wished to exclude the possibility of multiple courses we could have used *exclusive-or*, *cf.*, Rosen, p.5, instead of inclusive-or. So our antecedent about the main course would then read $B \oplus F$ or, equivalently, $(B \vee F) \wedge \neg(B \wedge F)$. The antecedent about the vegetable could be changed similarly. The deduction is still invalid with this formalization. ■

(b) Either John or Bill is telling the truth. Either Sam or Bill is lying. Thus, either John is telling the truth or Sam is lying.

Solution. We interpret "John is lying," to be the negation of "John is telling the truth." Similarly for the corresponding propositions involving Bill and Sam. The deduction is:

$$\frac{J \vee B, \quad \neg S \vee \neg B}{J \vee \neg S}$$

where

$$\begin{aligned} J &::= \text{“John is telling the truth.”} \\ B &::= \text{“Bill is telling the truth.”} \\ S &::= \text{“Sam is telling the truth.”} \end{aligned}$$

This deduction is valid. It is an example of a common "cancellation" or *cut* rule that lets us get rid of the proposition B in the conclusion. ■

(c) Either sales will go up and the boss will be happy, or expenses will go up and the boss won't be happy. Therefore, sales and expenses will not both go up.

Solution. The deduction is:

$$\frac{(S \wedge H) \vee (E \wedge \neg H)}{\neg(S \wedge E)}$$

where

$$\begin{aligned} S &::= \text{“Sales will go up.”} \\ H &::= \text{“The boss will be happy.”} \\ E &::= \text{“Expenses will go up.”} \end{aligned}$$

This deduction is invalid. For example, $S \wedge E \wedge H$ is consistent with the antecedent but not with the conclusion. ■

Problem 2. Are the following specifications[1] consistent?

1. If the file system is not locked, then
   (a) new messages will be queued.
   (b) new messages will be sent to the messages buffer.
   (c) the system is functioning normally, and conversely.
2. If new messages are not queued, then they will be sent to the messages buffer.
3. New messages will not be sent to the message buffer.

(a) Begin by translating the parts of the specification into propositional formulas using four propositional variables:

$$\begin{aligned} L &::= \text{file system locked}, \\ Q &::= \text{new messages are queued}, \\ B &::= \text{new messages are sent to the message buffer}, \\ N &::= \text{system functioning normally}. \end{aligned}$$

Solution. The translations of the specifications are:

$$\neg L \longrightarrow Q \qquad \text{(Spec. 1.(a))}$$
$$\neg L \longrightarrow B \qquad \text{(Spec. 1.(b))}$$
$$\neg L \longleftrightarrow N \qquad \text{(Spec. 1.(c))}$$
$$\neg Q \longrightarrow B \qquad \text{(Spec. 2.)}$$
$$\neg B \qquad \text{(Spec. 3.)}$$

■

(b) The specification is consistent if there is an assignment of truth values to the variables that makes every expression true. *Use a truth table* to determine whether the specification is consistent.

Solution. We can construct a truth table with sixteen lines—one for each way of assigning truth values to the four variables L, N, Q, and B. For each line, we could record the truth values of these five statements above.

If all five statements are true for some assignment of truth values to the variables, then the system is consistent. If for every one of the sixteeen possible truth assignments, at least one of the five statements is false, then the system is inconsistent. Carrying out the calculation shows that there is a unique assignment of True/False values to L, N, Q, and B (see below) that satisfies all the specifications. ■

[1] Rosen, Exercise 1.1.36

(c) Use *simple reasoning by cases* to find a truth assignment that confirms that this system specification is consistent. Explain why there is only one such assignment.

Solution. We can avoid the full truthtable calculation if we reason by cases.

Case 1 (B is True): Then the last formula, (Spec. 3.), is false, and the whole specification is false.

Case 2 (B is False): Now (Spec. 2.) and (Spec. 1.(b)) can be true only if Q and L are true. Since L is true, (Spec. 1.(c)) can be true only if N is false. Thus, we have deduced that in order to be consistent, we must have

$$\begin{aligned} L &= \text{True} \\ N &= \text{False} \\ Q &= \text{True} \\ B &= \text{False.} \end{aligned}$$

From the way this assignment was constructed, we know it ensures that formulas from (Spec. 1.(b)) on are true. So all that remains is to check formula (Spec. 1.(a)), and indeed it is also true under this assignment.

So the system is consistent, and this is the only assignment that will satisfy it. ■

Problem 3. [Optional] Suppose x is a real number. Prove **by cases** that there is a real y such that $\frac{y+1}{y-2} = x$ if and only if $x \neq 1$.

Solution. We consider the cases $x \neq 1$ and $x = 1$ separately.

Case 1 ($x \neq 1$): We need to prove that there exists $y \in \mathbb{R}$ such that $\frac{y+1}{y-2} = x$. In particular, consider $y = \frac{2x+1}{x-1}$ (derived by algebraic manipulation), which is a well-defined real number because $x \neq 1$. Then:

$$\begin{aligned} \frac{y+1}{y-2} &= \frac{((2x+1)/(x-1))+1}{((2x+1)/(x-1))-2} \\ &= \frac{\frac{2x+1+(x-1)}{x-1}}{\frac{2x+1-2(x-1)}{x-1}} \\ &= \frac{\frac{3x}{x-1}}{\frac{2x+1-2x+2}{x-1}} \\ &= \frac{3x}{3} \\ &= x. \end{aligned}$$

Therefore, (by the inference rule called *existential generalization*) $\exists y \in \mathbb{R}$ such that $\frac{y+1}{y-2} = x$.

Case 2 ($x = 1$): We need to prove that no $y \in \mathbb{R}$ exists such that $\frac{y+1}{y-2} = x$. Suppose, for contradiction, that such a y did exist. Then:

$$
\begin{aligned}
\frac{y+1}{y-2} = x \iff & \frac{y+1}{y-2} = 1 && \text{(because } x = 1\text{)} \\
\longrightarrow\ & y + 1 = y - 2 && \text{(multiply both sides by } y - 2\text{)} \\
\iff & 1 = -2 && \text{(subtract } y \text{ from both sides),}
\end{aligned}
$$

a contradiction. Therefore our supposition is wrong, and there is no y such that $\frac{y+1}{y-2} = x$.

■