Massachusetts Institute of Technology
6.042J/18.062J, Fall '02: Mathematics for Computer Science
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Solutions to In-Class Problems — Week 2, Fri

Problem 1. What amounts of postage can be made from 6¢ stamps and 10¢ stamps? Prove it.

Solution. 6¢, 10¢, 12¢, and every even number of cents ≥ 16¢.

We define our induction hypothesis $P(n)$ to be the following: an even amount n of postage can be assembled from 6¢ stamps and 10¢ stamps, for $n \geq 16$.

Cases $n = 16, 18, 20$¢ can be gotten explicitly. Then for even $n \geq 22$ we do the following: Assuming our hypothesis holds for $k \leq n$ we assemble $n - 4$¢ and add a 6¢ stamp to get $n + 2$ which is the next even number after n. We can't get an odd value of postage because 6 and 10 are both even. ■

Problem 2. Pinpoint, and illustrate with a counterexample, *exactly* where the following proof goes awry.

An integer, m, *divides* an integer, n, in symbols, $m \mid n$, iff there is an integer k such that $km = n$.

False Claim. *For any positive integers $p, x_1, x_2, \ldots, x_n$, if $p \mid x_1 x_2 \cdots x_n$, then $p \mid x_i$ for some i between 1 and n.*

False proof. (by strong induction on n.)

The induction hypothesis is the False Claim itself.

Base case ($n = 1$): When $n = 1$, we have $p \mid x_1$, therefore we can let $i = 1$ and conclude $p \mid x_i$.

Induction step: Now assuming the claim holds for all $k \leq n$, we must prove it for $n + 1$.

So suppose $p \mid x_1 x_2 \cdots x_{n+1}$. Let $y_n = x_n x_{n+1}$, so $x_1 x_2 \cdots x_{n+1} = x_1 x_2 \cdots x_{n-1} y_n$. Since the righthand side of this equality is a product of n terms, we have by induction that p divides one of them. If $p \mid x_i$ for some $i < n$, then we have the desired i. Otherwise $p \mid y_n$. But since y_n is a product of the two terms x_n, x_{n+1}, we have by strong induction that p divides one of them. So in this case $p \mid x_i$ for $i = n$ or $i = n + 1$. □

Solution. The statement "we have by strong induction that p divides one of them" is the place where the proof breaks down: it appeals to strong induction to justify applying the induction hypothesis for $2 = k \leq n$. But the base case was $n = 1$, so we can't assume $2 \leq n$. Note that the reasoning above is fine for every $n \geq 2$, so the whole proof would be fine if we had an argument to prove the claim for $n + 1 = 2$. But of course we can't prove it for $n + 1 = 2$ because it isn't true: let $p = 6, x_1 = 2, x_2 = 3$. Even though the reasoning is fine for every $n \geq 2$, the fact that it failed for $n = 1$ let us deduce a completely false result. ■

Problem 3. Here's a different proof that the square root of 2 is irrational, this one based on the Least Number Principle.

Proof. The proof is by contradiction.

Assume that the square root of 2 is rational. Then $\sqrt{2}$ is equal to m/n for some positive integers m and n. So n has the property that

$$n \text{ and } n\sqrt{2} \text{ are positive integers.} \tag{1}$$

Now by the Least Number Principle, there must be a *smallest* positive integer, n, satisfying (1). For this n, let $n_0 ::= n(\sqrt{2} - 1)$. But then n_0 is smaller than n, and n_0 and $n_0\sqrt{2}$ are positive integers, a contradiction. □

(a) Are you convinced by this proof? Briefly explain.

In particular, why is n_0 an integer?

Solution. Because it equals $n\sqrt{2} - n$ and n and $n\sqrt{2}$ are integers by hypothesis. ■

Why is n_0 positive?

Solution. Because $(\sqrt{2} - 1) \approx 0.414\ldots$ is positive. ■

Why is n_0 smaller than n?

Solution. Because $(\sqrt{2} - 1) < 0.415 < 1$. ■

Why is $n_0\sqrt{2}$ a positive integer?

Solution. It's an integer because $n_0\sqrt{2} = (n\sqrt{2} - n)\sqrt{2} = 2n - n\sqrt{2}$, which is again an integer because both terms are integers. It's positive because $2n - n\sqrt{2} = (2 - \sqrt{2})n > 0.58n > 0$. ■

We can generalize this proof to prove that $\sqrt{k}$ is irrational for integers k other than 2. Simply revise the next to last sentence to read, " … and let $n_0 ::= n(\sqrt{k} - \left\lfloor \sqrt{k} \right\rfloor)$". (For any real number, r, the expression $\lfloor r \rfloor$ denotes r rounded *down* to the nearest integer.)

(b) Are you convinced by this argument? Explain.

Solution. You shouldn't be convinced because the conclusion is wrong, For example, $\sqrt{k}$ is certainly rational if $k = 4$.

The proof works for $\sqrt{k}$ as long as k is not a perfect square. But if k is a perfect square, then the proof breaks down in the next to last sentence: the number $n_0(\sqrt{k} - \lfloor\sqrt{k}\rfloor)$ is not positive in this case, but zero. ■

Problem 4. We are given a chocolate bar with $m \times n$ squares of chocolate, and our task is to divide it into mn individual squares. We are only allowed to split one piece of chocolate at a time using a vertical or a horizontal break.

For example, suppose that the chocolate bar is 2×2. The first split makes two pieces, both 2×1. Each of these pieces requires one more split to form single squares. This gives a total of three splits.

(a) Use strong induction to conclude the following:

Theorem. *To divide up a chocolate bar with $m \times n$ squares, we need at most $mn - 1$ splits.*

Solution. This theorem does not immediately lend itself to an induction proof, since there are two variables. In general, propositions involving several natural-valued variables can often be proved by using a sort of nested induction. However, in this case, we can get by with a single-variable induction and a trick.

Intuitively, to break up a big chocolate bar, we need one split to make two pieces, and then we can break up the two pieces recursively. This suggests a proof using strong induction on the *size* of the chocolate bar, where size is measured in chocolate squares. Now instead of a problem involving two variables (the two dimensions), we have a problem in one variable (the size). With this simplification, we can prove the theorem using strong induction.

Proof. The proof is by strong induction on the size of the chocolate bar. Let $P(k)$ be the proposition that a chocolate bar of size k requires at most $k - 1$ splits.

Base case, $k = 1$: $P(1)$ is true because there is only a single square of chocolate, and $1 - 1 = 0$ splits are required.

Induction step: We suppose $k \geq 1$ and any chocolate bar of size s, where $1 \leq s \leq k$, requires at most $s - 1$ splits. We must now show there is a way to split a chocolate bar of size $k + 1$ with at most k splits.

To do this, first break the chocolate bar of size $k + 1$ into two smaller pieces of size p and q where $p + q = k + 1$. This is certainly possible because the size of the bar is at least two. Now the pieces of sizes p and q are between one and k, so by strong induction, breaking these two pieces into single squares requires only $p - 1$ and $q - 1$ splits, respectively. The total number of splits required to break the bar of size $k + 1$ into single squares is therefore at most $1 + (p - 1) + (q - 1) = p + q - 1 = (k + 1) - 1 = k$.

This shows that $P(k)$ implies $P(k + 1)$, and the claim is proved by strong induction. □

■

(b) The theorem proves that *at most* $mn - 1$ splits are needed to divide up an m by n chocolate bar. Could you do better?

Solution. The proof really shows that exactly $s-1$ splits are needed. Observe that in the induction step, no matter how we split the bar into smaller bars of size p and q, exactly $(p-1)+(q-1) = s-1$ splits are required. ■