Massachusetts Institute of Technology
6.042J/18.062J, Fall '02: Mathematics for Computer Science
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In-Class Problems — Week 7, Fri

1 Problems

Problem 1. There is a number a such that $\sum_{i=1}^{\infty} i^p$ converges iff $p < a$. What is the value of a? Prove it.

Problem 2. Suppose $f_1, f_2, g_1, g_2 : \mathbb{N} \to \mathbb{N}$. Prove the following:

(a) If $f_1 = o(g_1)$, then $f_1 + g_1 \sim g_1$.

(b) If $f_1 = o(g_1)$ and $f_2 = o(g_2)$, then $f_1 + f_2 = o(g_1 + g_2)$.

(c) If $f_1 = O(g_1)$ and $f_2 = O(g_2)$, then $f_1 + f_2 = O(g_1 + g_2)$.

(d) If $f_1 \sim g_1$ and $f_1 \geq f_2 \sim g_2 \leq g_1$, then it is *not* necessarily true that $f_1 - f_2 \sim g_1 - g_2$.

Problem 3. Indicate which of the following holds for each pair of functions $(f(n), g(n))$ in the table below. Assume $k \geq 1$, $\epsilon > 0$, and $c > 1$ are constants.

$f(n)$	$g(n)$	$f = O(g)$	$f = o(g)$	$g = O(f)$	$g = o(f)$	$f = \Theta(g)$	$f \sim g$
2^n	$2^{n/2}$						
$\sqrt{n}$	$n^{\sin n\pi/2}$						
$\log(n!)$	$\log(n^n)$						
n^k	c^n						
$\log^k n$	n^ϵ						

Problem 4. It is a standard fallacy to think that given n quantities each of which is $O(1)$, their sum would have to be $O(n)$. In fact, such a sum can grow arbitrarily fast:

Let $g : \mathbb{N} \to \mathbb{N}^+$ be *any* function. Explain how to define a sequence $f_1, f_2, \ldots$ of functions from $\mathbb{N}$ to $\mathbb{N}$ such that

$$f_i(n) = O(1) \tag{1}$$

for all $i \geq 1$, but

$$\sum_{i=1}^{n} f_i(n) \neq O(g(n)). \tag{2}$$

2 Appendix: Asymptotic Notations

Definition 4.1. For functions $f, g : \mathbb{R} \to \mathbb{R}$, we say f is *asymptotically equal* to g, in symbols,

$$f(x) \sim g(x)$$

iff

$$\lim_{x\to\infty} f(x)/g(x) = 1.$$

Definition 4.2. For functions $f, g : \mathbb{R} \to \mathbb{R}$, we say f is *asymptotically smaller* than g, in symbols,

$$f(x) = o(g(x)),$$

iff

$$\lim_{x\to\infty} f(x)/g(x) = 0.$$

Definition 4.3 (Using Limits). Given functions $f, g : \mathbb{R} \mapsto \mathbb{R}$, with g nonnegative, we say that[1]

$$f = O(g)$$

iff

$$\limsup_{x\to\infty} |f(x)| / g(x) < \infty.$$

Definition 4.4 (Standard). Given functions $f, g : \mathbb{R} \mapsto \mathbb{R}$, we say that

$$f = O(g)$$

iff there exists a constant $c \geq 0$ and an x_0 such that for all $x \geq x_0$, $|f(x)| \leq cg(x)$.

Definition 4.5.

$$f = \Theta(g) \quad \text{iff} \quad f = O(g) \wedge g = O(f).$$

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[1]

$$\limsup_{x\to\infty} h(x) ::= \lim_{x\to\infty} \text{lub}_{y\geq x} h(y).$$