Solutions to In-Class Problems — Week 10, Mon

Problem 1. **(a)** What is the probability that a random poker hand contains cards from at most two suits?

Solution. The sample space consists of the $\binom{52}{5}$ possible hands. Each hand is equally likely and comes up with probability $1/\binom{52}{5}$. Therefore, the probability that a hand contains at most two suits is equal to the number of different hands of this type divided by $\binom{52}{5}$. All that remains is to count the number of poker hands with at most two suits.

There are $\binom{4}{2}$ ways to choose two suits and $\binom{26}{5}$ ways to choose five cards from these two suits. However, this triple-counts hands containing cards from a single suit. (For example, a hand with all spades is counted once as a spades-hearts hand, a second time as a spades-clubs hand, and a third time as a spades-diamonds hand.) The number of hands with a single suit is $4 \cdot \binom{13}{5}$. Therefore, the number of hands with at most two suits is

$$\binom{4}{2} \cdot \binom{26}{5} - 2 \cdot 4 \cdot \binom{13}{5},$$

and the probability that a random poker hand contains cards from at most two suits is:

$$\frac{\binom{4}{2} \cdot \binom{26}{5} - 2 \cdot 4 \cdot \binom{13}{5}}{\binom{52}{5}} \approx 0.15$$

■

(b) Suppose you repeatedly flip a fair coin (*fair* means that heads and tails are equally likely to appear). What is the probability that you flip exactly 4 heads out of 8? between 3 and 5 heads?

Solution. The probability of exactly m heads in n tosses is $\binom{n}{m}/2^n$. So the probability of exactly 4 heads is:

$$\frac{\binom{8}{4}}{2^8} = \frac{8 \cdot 7 \cdot 6 \cdot 5}{4 \cdot 3 \cdot 2 \cdot 1 \cdot 2^8} = \frac{35}{128} \approx 0.27.$$

Similarly, the probability of between 3 and 5 heads is:

$$\frac{\binom{8}{3} + \binom{8}{4} + \binom{8}{5}}{2^8} \approx 0.60.$$

■

Problem 2. Here is a coin-flipping game: flip a fair coin at most four times but stop sooner if two Heads come up in a row.

(a) What is a suitable sample space (set of outcomes) for this game?

Solution. The sample points are:

$$TTTT, TTTH, TTHT, TTHH, THTT, THTH, THH, HTTT, HTTH, HTHT, HTHH, HH$$

which we obtained by drawing a tree (not shown) illustrating the outcomes after each flip. ■

(b) What are reasonable probabilities to assign to the sample points?

Solution. Each node of the tree that is not a leaf has two branches describing the outcomes Head and Tail, and these are equally likely. So the probability of any final pattern, s, of Heads and Tails is

$$\Pr\{s\} ::= \frac{1}{2^{|s|}},$$

where $|s|$ is the length of s. ■

(c) You win when two Heads in a row have come up. Are the odds of winning this game in your favor?

Solution. Winning points are HH, THH, TTHH, and HTHH so

$$\Pr\{\text{win}\} = \frac{1}{2^2} + \frac{1}{2^3} + 2 \cdot \frac{1}{2^4} + = \frac{4 + 2 + 2 \cdot 1}{16} = \frac{1}{2}.$$

So the odds of losing are the same as those of winning. ■

Problem 3. Suppose the game *Let's Make A Deal* is changed slightly. Instead of having 3 doors with 1 grand prize, in the new game there are 4 doors with 2 grand prizes. Otherwise, the rules of the game remain the same:

1. The contestant picks one of the doors.
2. Assistant Carol opens a different door that has a goat behind it.
3. The contestant can then **stick** with his original pick or **switch** to an unopened door. He wins a prize only if his final pick is the door with the prize.

Assume that the prizes are equally likely to be placed behind each of the doors, and that Carol and the contestant are also equally likely to pick each door among their possible choices.

(a) In the new game, what is the probability of winning with the "stick" strategy? How about the "switch" strategy?

Solution. The player wins with a "stick" strategy iff he chooses the "correct door" in the first place. Therefore the probability of winning with the "stick" strategy is $1/2$.

To analyze the switch strategy we form the usual tree, as shown in Figure (1).

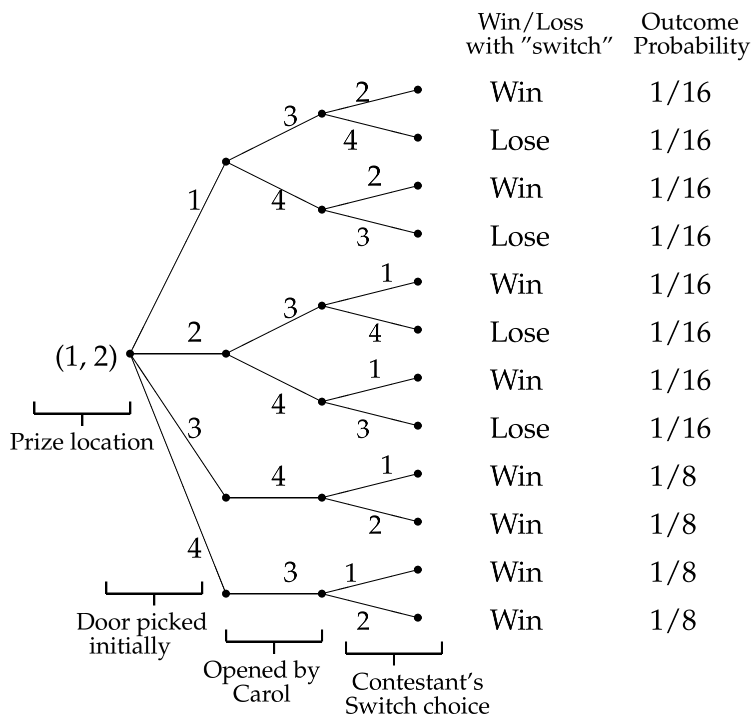

Figure 1: Coding Numbered Trees

By symmetry, we may assume that the prizes are behind doors 1 & 2. The possible outcomes are then of the form:

(contestant-initial-choice, Carol-choice, contestant-switch-choice) with probabilities:

doors	probability	win-with-switch
(1,3,2)	1/16	win
(1,3,4)	1/16	lose
(1,4,2)	1/16	win
(1,4,3)	1/16	lose
(2,3,1)	1/16	win
(2,3,4)	1/16	lose
(2,4,1)	1/16	win
(2,4,3)	1/16	lose
(3,4,1)	1/8	win
(3,4,2)	1/8	win
(4,3,1)	1/8	win
(4,3,2)	1/8	win

So the switch strategy wins 3/4 of the time. ■

(b) In the original Monty Hall game, the probabilities of the winning-with-stick and the winning-with-switch strategies summed to one, but here they don't. How come?

Solution. In the original Monty Hall game with one prize, the switch strategy would win iff the stick strategy would not win, so winning-with-stick and winning-with-switch were complementary events whose probabilities had to sum to one.

This time the two probabilities do not need to sum up to 1 because they are no longer disjoint events. It is possible for both the "switch" and the "stick" strategy to pick doors that both have a prize behind them, because there are two prizes. So the same outcome can be a win for both strategies. ■