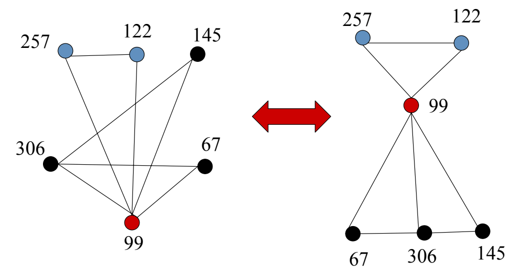
Mathematics for Computer Science

MIT 6.042J/18.062J

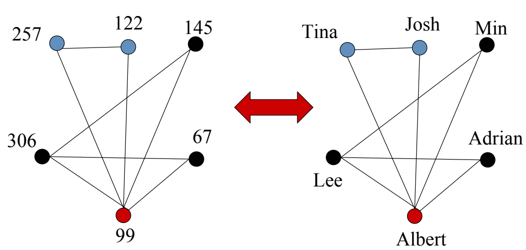
More Graphs

Prof. Albert Meyer & Dr. Radhika Nagpal

Topology, not Geometry



Equivalent (Isomorphic) Graphs

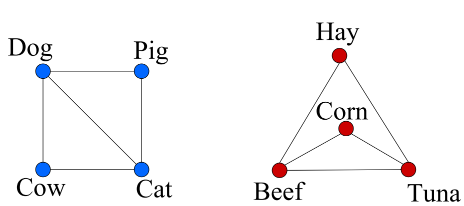


Graph Isomorphism

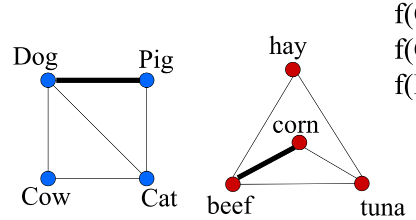
Graphs G_1 and G_2 are isomorphic if
there exists a bijection $f: V_1 \rightarrow V_2$ such that
for all u,v in V_1
– the **edge (u,v)** is in G_1
– *iff* the **edge (f(u), f(v))** is in G_2

- If there is a one to one correspondence between the nodes of G_1 and G_2 that preserves all edge connections.

Are these Isomorphic?



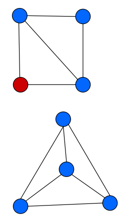
Find a Mapping



Function
f(Dog) = beef
f(Cat) = tuna
f(Cow) = hay
f(Pig) = corn

Finding the Mapping

- Not easy, can try all possible mappings
  - Roughly n! possibilities
- Can test for Invariants
  - Same number of nodes, edges
  - Same degree distributions
  - Preserves cycles, longest path, etc

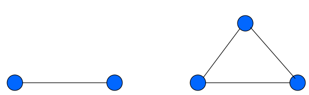


In-class Problem 1

Problem with False Proof 1

Proof (silently) assumes any 2-ended G_{n+1} can be built from a 2-ended G_n. This isn't true!

Consider the counter-example, it is two ended but I cannot construct it by adding an edge to another two ended graph.

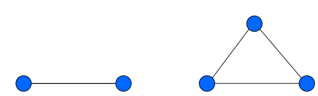


Problem with False Proof 2

After removing a vertex from G_{n+1}, the claim that G_n still has 2 vertices of degree 1 and rest degree 2

This is not true!

The same counter example works:



Revisit: Coloring with d_{max} colors

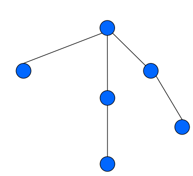
- Induction Hypothesis

  P(n) = a graph with *n* vertices and maximum degree d_{max} can be colored with $d_{max} + 1$ colors

- Inductive Step
  - Do you justify why your proof doesn't have the same pitfalls?

Trees

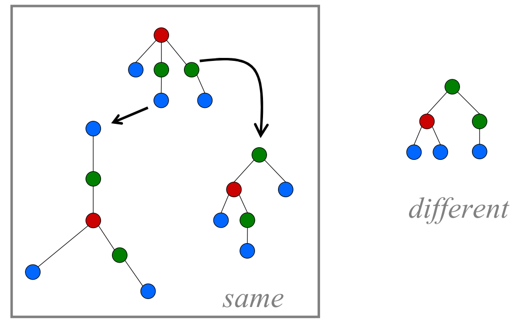
- *Definition:* A tree is simple connected graph with no cycles.



Applications of Trees

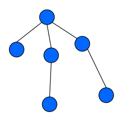
- Data structures for sorting, searching
- Spanning Trees
- Game Trees (alpha-beta trees)
- Prefix codes (Huffman encoding)
- Many algorithms based on trees (6.046)

Tree Isomorphisms



Trees

- *Definition:* A tree is a simple connected graph with no cycles.



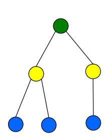
- ***Exercise*:** Draw a tree with 5 vertices
- ***Question:*** How many edges does your tree have? 3,4 or 5?

Another Tree Definition

- No matter how you draw it, you get 4 edges.
- *Definition 2:* A tree is a connected graph with n vertices and $n - 1$ edges.
- In fact, *a tree is the smallest connected graph on n vertices!*

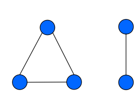
Equivalent Definitions of Trees

- A connected graph with no cycles
- A connected graph where $|E| = |V| - 1$
- A connected graph where removing any edge leaves a disconnected graph
- A graph such that there exists a unique simple path between any two vertices

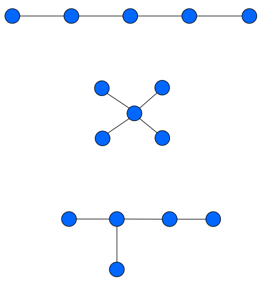


Be careful with these definitions

- What is wrong with this definition?
  - A tree is a graph with n vertices and $n - 1$ edges.
- Counter-example



Different Trees with 5 vertices



Vertex degrees

1 2 2 2 1
4 1 1 1 1
1 3 1 2 1

Sum is always 8
(2 × edges)

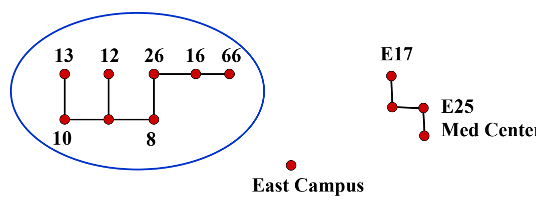
MIT Building Connections

Connectivity and Paths

- Can you get from building 10 to 36 without crossing more than 5 other buildings
  - Is there a path of length *k* from *u* to *v*?
- How many different ways are there to get from building 10 to building 36?
  - How many different paths are there from *u* to *v*?

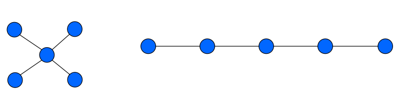
Connected Components

- Can we get from building 10 to building E17?
  - Is there a path between *u* and *v* ?
  - Are *u* and *v* connected?



Smallest Connected Graph

- MIT administration wants the number of physical connections between buildings to be minimum but still have everything connected.
  - What is the smallest connected graph I can construct? ANY TREE



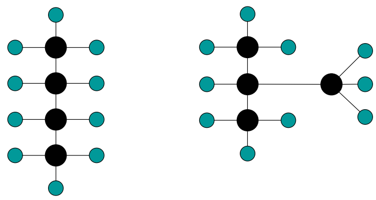
Cut Edge

- *Definition:* An edge is a **cut edge** if removing it from the graph disconnects two connected components

Problem with our smallest connected graph
  - any edge disruption disconnects the graph!

Graphs with Same Degrees

- Example: isomers of butane
  - Same degree distribution, but not isomorphic



Why are some problems easier than others?

- 2 colorable (no odd cycles)
- 3 colorable (NP- complete)

- Euler circuits (connected and all even degree)
- Hamiltonian circuits (NP complete)