

Problem Set 8

Reading: [Week 8 Notes](#), Optional: Rosen §4.1–2

Problem 1. Write closed form formulas which may involve factorials for the following functions:

- (a) $G(n) ::=$ the number of directed graphs with vertices $\{1, 2, \dots, n\}$ (note, self-loops are permitted in directed graphs).
- (b) $U(n) ::=$ the number of directed graphs with vertices $\{1, 2, \dots, n\}$ such that if the edge (i, j) is in the graph then (j, i) can not be in the graph, where i and j are distinct vertices.
- (c) $M(n) ::=$ the number of functions from $\{1, 2, \dots, n\}$ to $\{1, 2, \dots, 2n\}$.
- (d) $B(n) ::=$ the number of *injections* from $\{1, 2, \dots, n\}$ to $\{1, 2, \dots, 2n\}$.
- (e) Order the above functions so that each function is $O()$ of the function to its right. Also indicate whether the $\Theta()$ relationship holds between any successive pair of functions. No explanation necessary.

Problem 2. How many sequences of 6-digits are there that do not contain “123” or “456”? For example, “000456,” “112397,” “456123,” do contain one or both of these 3-digit patterns, but “645111” and “112233” do not.

Problem 3. The Towers of Hanoi game involves stacking 32 disks onto three vertical pegs, *cf.*, Rosen §5.1, Example 5. Each disk has a hole in the middle through which a peg can slide, and all the disks have different diameters. An arrangement of the disks on the pegs is allowed only if no disk rests on a smaller disk. How many different such arrangements are allowed?

Problem 4. Suppose we are given a list of n integers a_1, \dots, a_n , which need not be distinct. Prove that there is always a set of consecutive numbers $a_{k+1}, a_{k+2}, \dots, a_l$ whose sum $\sum_{i=k+1}^l a_i$ is a multiple of n .

Problem 5. A positive integer is called *square-free* if it is not divisible by the square of any positive integer greater than 1. For example $35 = 5 \cdot 7$ is square-free but $18 = 2 \cdot 3^2$ is not. 1 is square-free. Use inclusion-exclusion to find the number of square-free positive integers strictly less than 201.

