

## Solutions to In-Class Problems — Week 7, Wed

**Problem 1.** Suppose you were about to enter college today and a college loan officer offered you the following deal: \$25,000 at the start of each year for four years to pay for your college tuition and an option of choosing one of the following repayment plans:

**Plan A:** Wait four years, then repay \$20,000 at the start of each year for the next ten years.

**Plan B:** Wait five years, then repay \$30,000 at the start of each year for the next five years.

Suppose the annual interest rate paid by banks is 7% and does not change in the future.

(a) Assuming that it's no hardship for you to meet the terms of either payback plan, which one is a better deal? (You will need a calculator.)

**Solution.** \$1 today will be worth \$1.07 next year, and \$1.07<sup>2</sup> the year after, *etc.* So set  $r = \frac{1}{1.07}$ . Then:

current value of Plan A

$$\begin{aligned} &= \sum_{y=4}^{13} 20000 \cdot r^y \\ &= \sum_{y=0}^9 20000 \cdot r^{y+4} \\ &= r^4 \cdot \sum_{y=0}^9 20000 \cdot r^y \\ &= 20000r^4 \cdot \sum_{y=0}^9 r^y \\ &= 20000r^4 \cdot \frac{1 - r^{10}}{1 - r} \\ &= \$114,66.69 \end{aligned}$$

current value of Plan B

$$\begin{aligned}
 &= \sum_{y=5}^9 30000 \cdot r^y \\
 &= \sum_{y=0}^4 30000 \cdot r^{y+5} \\
 &= r^5 \cdot \sum_{y=0}^4 30000 \cdot r^y \\
 &= 30000r^5 \cdot \sum_{y=0}^4 r^y \\
 &= 30000r^5 \cdot \frac{1-r^5}{1-r} \\
 &= \$93,840.63
 \end{aligned}$$

You should clearly take Plan B. You will be paying back much less in today's dollars. ■

(b) What is the loan officer's effective profit (in today's dollars) on the loan?

**Solution.** The value of the money you are given is:

$$\begin{aligned}
 \text{Loan} &= \sum_{y=0}^3 25000 \cdot r^y \\
 &= 25000 \cdot \sum_{y=0}^3 r^y \\
 &= 25000 \cdot \frac{1-r^4}{1-r} \\
 &= \$90,607.90
 \end{aligned}$$

Therefore, the loan officer's profit is effectively \$3,233. (Or \$24,059 if we are not on the ball). ■

**Problem 2.** We begin with two large glasses. The first glass contains a pint of water, and the second contains a pint of wine. We pour  $1/3$  of a pint from the first glass into the second, stir up the wine/water mixture in the second glass, and then pour  $1/3$  of a pint of the mix back into the first glass and repeat this pouring back-and-forth process a total of  $n$  times.

(a) Describe a closed form formula for the amount of wine in the first glass after  $n$  back-and-forth pourings.

**Solution.** The state of the system of glasses/wine/water at the beginning of a round of pouring and pouring back is determined by the total amount of wine in the first glass. Suppose at the beginning of some round, the first glass contains  $w$  pints of wine,  $0 \leq w \leq 1$  and  $1 - w$  pints of water. The second glass contains the rest of the wine and water.

Pouring  $1/3$  pint from first glass to second leaves  $2/3$  pints of liquid and  $(2/3)w$  wine in the first glass, and  $4/3$  pints of liquid and  $1 - (2/3)w$  wine in the second glass. Pouring  $1/3$  pint back from second into first transfers a proportion of  $(1/3)/(4/3)$  of the wine in the second glass into the first. So the round completes with both glasses containing a pint of liquid, and the first glass containing

$$(2/3)w + (1/4)(1 - (2/3)w) = 1/4 + w/2$$

pints of wine. After one more round, the first glass contains

$$1/4 + (1/4 + w/2)/2 = 1/4 + 1/8 + w/2^2$$

pints of wine, and after  $n$  more rounds

$$\begin{aligned} w/2^n + \sum_{i=1}^n (1/2)^{i+1} &= w/2^n + (1/2)\sum_{i=1}^n (1/2)^i \\ &= w/2^n + (1/2)(-1 + \sum_{i=0}^n (1/2)^i) \\ &= w/2^n + (1/2)(-1 + (1 - (1/2)^{n+1})/(1 - 1/2)) \\ &= w/2^n - 1/2 + 1 - (1/2)^{n+1} \\ &= w/2^n + 1/2 - (1/2)^{n+1}. \end{aligned}$$

Since  $w = 0$  initially, the pints of wine in the first glass after  $n$  rounds is

$$1/2 - (1/2)^{n+1}.$$

■

(b) What is the limit of the amount of wine in each glass as  $n$  approaches infinity?

**Solution.** The limiting amount of wine in the first glass approaches  $1/2$  from below as  $n$  approaches infinity. In fact, it approaches  $1/2$  no matter how the wine was initially distributed. This of course is what you would expect: after a thorough mixing the glasses should contain essentially the same amount of wine. ■

**Problem 3.** Using the method described in lecture, a truck can travel across any size desert if there is a large enough supply of gas at the border of the desert. Show that if there is a large enough supply of gas at the border, a truck can also make a *round trip* across any size desert.

**Solution.** Let  $R_n$  be the distance a truck can travel into the desert and return on  $n$  tanks of gas. Clearly,  $R_1 = 1/2$ .

On  $n + 1$  tanks, the truck will travel distance  $x$  and back  $n$  times, leaving  $1 - 2x$  tanks of gas at distance  $x$  into the desert on each trip. It then makes one more one-way trip to  $x$ . This leaves it

with  $n(1 - 2x) + 1 - x$  tank of gas at position  $x$ . Leaving an  $x$ th of a tank so it can get back, if the remaining  $(n(1 - 2x) + 1 - x) - x = (n + 1)(1 - 2x)$  tanks equal  $n$ , it can execute the  $n$ -tank round trip strategy from position  $x$  and still return to the desert border. So

$$(n + 1)(1 - 2x) = n \tag{1}$$

$$x = 1/2(n + 1) \tag{2}$$

$$R_{n+1} = R_n + x = R_n + 1/2(n + 1). \tag{3}$$

So

$$R_n = 1/2(1 + 1/2 + 1/3 + \cdots + 1/n) = H_n/2.$$

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