



Deviation from the Mean



Bernoulli Trials



$R ::=$ number of heads in n tosses of a fair coin

$$\Pr\{R = k\} = \binom{n}{k} 2^{-n} \quad \text{Binomial Distribution (fair)}$$

$$E[R] = \frac{n}{2}$$



Exactly the Mean?



What is $\Pr\{\textit{exactly } n/2 \textit{ heads}\}$?

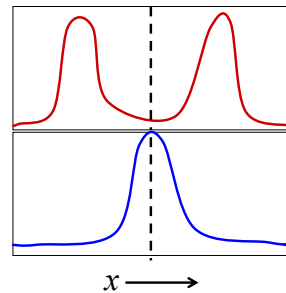
$$\Pr\{R = n/2\} = \binom{n}{n/2} 2^{-n}$$

If $n=100$, $\Pr\{R = 50\} = 8\% !!!$



Two Distributions, Same Mean

$\Pr\{R = x\}$



Two Dice with the Same Mean

Fair Dice

- $E[R] = 3.5$



Loaded Dice: (throws either 1 or 6)

- $E[R] = (1+6)/2 = 3.5!!!$



Noisy Transmission Line

Total number of bits transmitted = 10,000

$R =$ number of errors in 10,000 bits

$E[R] = 100$ bits



- But what does the error look like on any specific transmission? $\pm 100?$ $\pm 1000?$



What to Expect from “Expected Value”?

- If I toss n coins today, what should I expect to see?
- *Power of Probability*: the ability to predict the outcome of a specific event.



Back to Bernoulli Trials



EXPERIMENT:

Toss **100 coins** and count # of heads (R)

Run this experiment many times

- **Average** number of heads = ?
- **Exactly 50** heads = ?
- **Between 40-60** heads = ?



Scheme Buffer (of course!)

```

; THROW A SEQUENCE OF n TOSSES of an UNBIASED COIN
(define (ntoss n)
  (if (= n 0)
      '()
      (cons (random 2)
            (ntoss (- n 1)))))

; RUN the EXPERIMENT t TIMES, and return a list with # HEADS
(define (experiment n t)
  (if (= t 0)
      '()
      (cons (reduce + 0 (ntoss n)) ; count # of heads (ones)
            (experiment n (- t 1)))))

; Functions to compute average, exactly V heads, heads in RANGE
(define (average-heads foo) (exact->inexact (/ (reduce + 0 foo) (length foo))))
(define (total-heads-v foo v)(reduce + 0 (map (lambda (heads)(if (= heads v) 1 0)) foo)))
(define (total-heads-range foo low high) (reduce + 0 (map (lambda (heads)
  (if (and (<= heads high) (>= heads low)) 1 0)) foo)))

```



Results



EXPERIMENT:

Toss **100 coins** and count # of heads (R)

Ran this experiment 100 times

- **Average** number of heads = **49.5**
- **Exactly 50** heads = **4** out of 100
- **Between 40-60** heads = **97** out of a 100!



Giving *Meaning* to the “Mean”

Let $\mu ::= E[R]$

- What is $\Pr\{R \text{ is within } \mu \pm 10\}$?
- What is $\Pr\{R > 2\% \text{ error}\}$?
- On average, how much does R deviate from the expected value?
– What is $E[|R - \mu|]$?



Deviation from the Mean

Using the expected value to infer the probability of occurrence

- Markov Bound
- Chebyshev Bound (Variance)
- Chernoff Bound
- Binomial Approximation
- Actual Distribution...

More information about the distribution

6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

Sampling and Deviation from Mean

Using *event occurrence* to infer the expected value.

- Weak Law of Large Numbers
- Pairwise Independent Sampling Theorem
- Applications to Polling

6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

Intuition about Averages

- Lets say we test n people and find out that the average IQ = 100
- **EXERCISE:** What fraction of the people can have an IQ ≥ 200 ?

6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

IQ Higher than 200

Let f be the **fraction of people** who have an IQ of at least 200

- Then these people contribute at least $200f$ to the average IQ.
- So, $200f \leq 100$ or $f \leq 100/200 = \frac{1}{2}$
- No more than $\frac{1}{2}$ people have IQ ≥ 200 .

$$\Pr \{IQ \geq \text{value}\} \leq E[IQ] / \text{value}$$

6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

Markov Bound

If R is non-negative, then

$$\Pr \{R \geq x\} \leq \frac{E[R]}{x}$$

for $x \geq 0$.

6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

Another form of Markov Bound

Let $x = c \cdot E[R]$

$$\Pr \{R \geq c \cdot E[R]\} \leq 1/c$$

- Probability that deviate more than 2 times the expected value, is less than $\frac{1}{2}$

6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

Formal Proof of Markov Bound

$$\begin{aligned} E[R] &::= \sum_{r \in \text{range}(R)} r \cdot \Pr\{R=r\} \\ &= \sum_{r \geq x} r \cdot \Pr\{R=r\} + \sum_{0 \leq r < x} r \cdot \Pr\{R=r\} \\ &\geq \sum_{r \geq x} r \cdot \Pr\{R=r\} + 0 \\ &\geq \sum_{r \geq x} x \cdot \Pr\{R=r\} \\ &= x \cdot \sum_{r \geq x} \Pr\{R=r\} = x \cdot \Pr\{R \geq x\} \end{aligned}$$

6	9	13	7
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Markov Bound

Markov's bound is

- Weak
- Obvious

Turns out to be useful anyway.

6	9	13	7
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Better Bound

- Let $\mu ::= E[R]$
- What is $\Pr \{|R - \mu| \geq x\}$?
 - what is the probability that R deviates more than x from the expected value

6	9	13	7
12	10	5	
3	1	4	11
15	8	14	2

Chebyshev Bound

$$\Pr \{|R - \mu| \geq x\} \leq \frac{E[(R - \mu)^2]}{x^2}$$

Variance

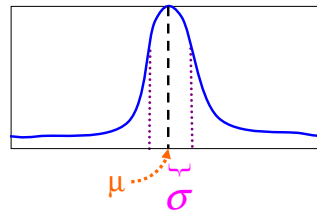
Probability that I see a value more than x away from the expected value

6	9	13	7
12	10	5	
3	1	4	11
15	8	14	2

Variance and Standard Deviation

$$\text{Var}[R] ::= E[(R - \mu)^2]$$

$$\sigma ::= \sqrt{\text{Var}[R]}$$



Why not just use $E[|R - \mu|]$?

6	9	13	7
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15	8	14	2

Problem 1

Write solutions on the board

6	9	13	7
12	10	5	
3	1	4	11
15	8	14	2

Interpreting Variance

Gambling Game: Same $E[\text{Win}]$
Different $\text{Var}[\text{Win}]$
(σ^2)

From Chebyshev:

$$\Pr\{|W - E[W]| \geq c\} \leq \text{Var}[W]/c^2$$

$$\text{so, } \Pr\{|W - E[W]| \geq 2\sigma\} \leq 1/4 = 25\%$$

$$\Pr\{W \text{ in within } \pm 2\sigma\} \geq 75\%$$



Conventional Wisdom

Probability that you are

within $\mu \pm 2\sigma > 75\%$

within $\mu \pm 3\sigma > 90\%$



Bernoulli Trials



$E[R] = \mu = 50$ heads

• $\Pr\{R \geq 60 \text{ heads}\} \leq 50/60$ (*Markov*)

$\text{Var}[R] = 25, \sigma = 5$

• $\Pr\{40 \leq R \leq 60\} \geq 1 - 25/100$ (*Chebyshev*)
= 75%

Actual Data = 97/100 times



In Class Problems

Problems 2-4