Mathematics for Computer Science

MIT 6.042J/18.062J

Combinatorics I.3

Trees, Bijections,
& Permutations

Counting Techniques

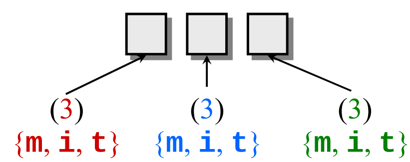
- Bijections
- Sum Rule, Inclusion-Exclusion
- Product Rule
- Pigeonhole Principle
- Trees
- Permutations

Counting Strings

How many strings of length 3 can be made from the alphabet

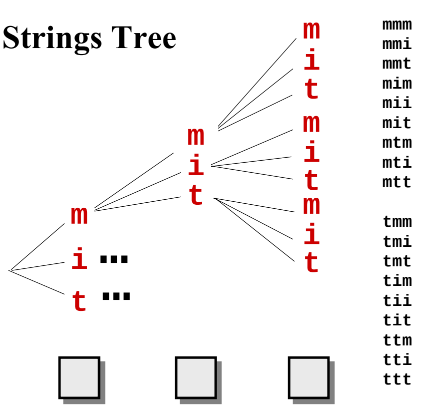
A = {`m`, `i`, `t`} ?

Counting Strings



Number of strings = 3·3·3 = 27

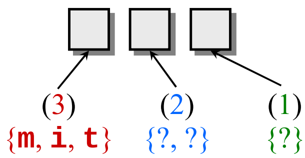
Strings Tree



Counting Rearrangements

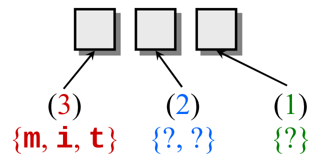
How many strings of length 3 can be made by *rearranging* the letters `mit`?

Permutations



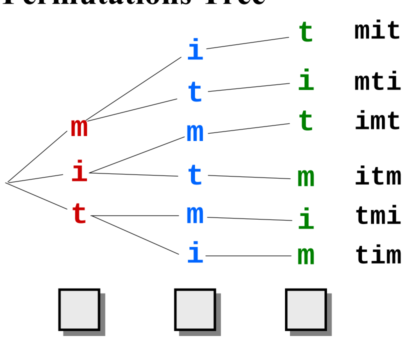
Number of strings = |*S*| = 3·2·1 = 6

Permutations



S = {**mit**, **mti**,
imt, **itm**,
tmi, **tim**}

Permutations Tree



Comparison

			perms	*strings*
n	n^2	2^n	$n!$	n^n
1	1	2	1	1
5	25	32	120	3125
10	100	1024	10^7	10^{10}
20	400	10^6	10^{18}	10^{26}
40	1600	10^{12}	10^{47}	10^{64}
100	10^4	10^{30}	10^{157}	10^{200}
1000	10^6	10^{300}	*Scheme dies*	

In-Class Problems

Problem 1

Important Technique: Matching

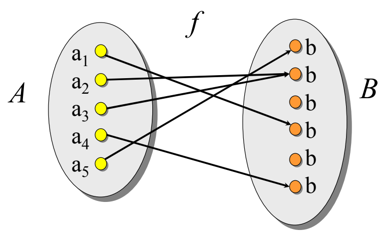
Make a bijection from the the set we wish to count and a set of known cardinality (or at least one we know how to count…).

Examples:

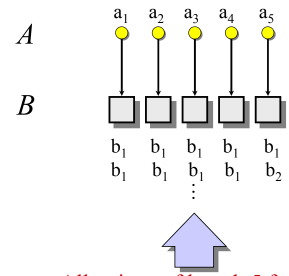
Functions	↔	Strings
Lists	↔	Trees
Graphs, Relations	↔	Matrices
Strings, Polynomials	↔	Balls-and-Bins

Counting Functions

- How many different functions are there from *A* to *B*?

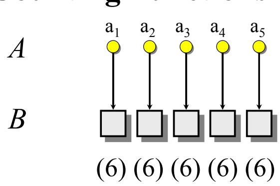


Mapping: Functions as Strings



f All strings of length 5 from the alphabet $\{b_1, b_2, \ldots, b_6\}$

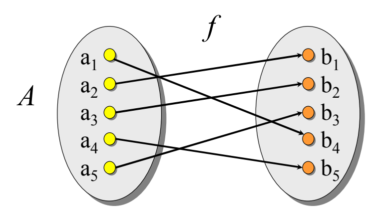
Counting Functions



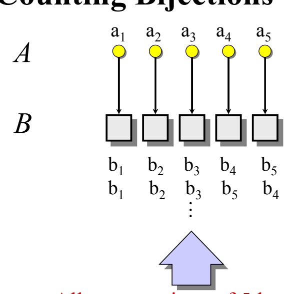
Total number of possible functions
= total number of possible strings
$= 6 \cdot 6 \cdot 6 \cdot 6 \cdot 6 = \boxed{6^5}$

Counting Bijections

- How many different bijections are there from *A* to *B*?

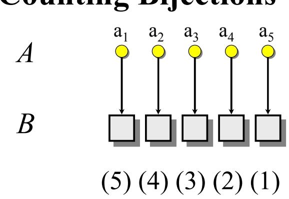


Counting Bijections



f All permutations of 5 letters from the alphabet $\{b_1, b_2, \ldots, b_5\}$

Counting Bijections



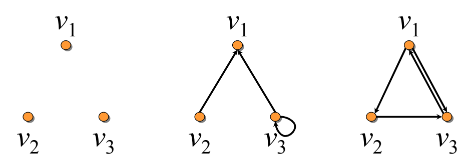
Total number of possible functions
= total number of possible permutations
$= 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = \boxed{5!}$

In-Class Problems

Problem 2

Counting Graphs

How many different directed graphs on n labeled nodes are there?



Graphs as Binary Strings

Maximum # edges = n^2

	e_1	e_2	e_3	e_4	…	e_{n^2}
G_1	1	0	1	0	…	1
G_2	0	1	1	0	…	1
G_3			⋮			
⋮						

Each string defines a different graph

2^{n^2} digraphs

Graphs as Boolean Matrices

	1	2	3	⋯	n
1	1	0	1		
2	0	1			
3	0	1			
⋮					
n					

Each boolean matrix defines a different graph

2^{n^2} digraphs

Proving That a Mapping Is a Bijection

To prove that $f : A \to B$ is bijective,

1. show that f maps *all* elements of A to *distinct* elements of B
   - *i.e.*, f is one-to-one or injective;

AND

2. show that for *every* element of B, f maps *some* element of A to it
   - *i.e.*, f is onto or surjective.

Proving That Two Sets Have the Same Size

To prove that $|A| = |B|$,

1. exhibit an injective function $f : A \to B$ and a surjective function $g : A \to B$, or
2. exhibit an injective function $f : A \to B$ and an injective function $g : B \to A$, or
3. exhibit a surjective function $f : A \to B$ and a surjective function $g : B \to A$.

Example: Graph Isomorphism

G_1 and G_2 are ***isomorphic*** if there is a bijection from the nodes of G_1 to the nodes of G_2, that preserves adjacency.

- *Question:* How many bijections might we need to check to determine whether G_1 and G_2 are isomorphic?
- *Answer:* All possible bijections from the vertices of G_1 to $G_2 = n!$

In-Class Problems

Problems 3-4