Massachusetts Institute of Technology
6.042J/18.062J, Fall '02: Mathematics for Computer Science
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Problem Set 6-7

Reading: Week 6 Notes, Optional: Rosen §3.3; Week 7 Notes, Optional: Rosen §1.7–8.

Problem 1. We can generalize win-lose 2-player terminating games of perfect information to games with "payoff" amounts. In these games, two players called the *max-player* and the *min-player* alternate moves until the game ends with the min-player paying some payoff amount to the max-player. How much the min-player pays depends on how the game ends. Negative payoffs mean the max-player pays the min-player. The max-player moves first.

Such games are defined by finite-path trees with leaves labelled with real numbers. These are the payoff amounts. The max-player tries to arrive at a leaf with as large a payoff as possible, and the min-player tries to minimize the payoff to the max-player.

Definition. The set of payoff-game trees, PayT, can be defined recursively as follows:

1. If T is a graph with one vertex, v, and no edges, then T is a PayT and $\text{root}(T) ::= v$.
2. If $\mathcal{S}$ is a set of PayT's such that no vertex occurs in more than one tree in $\mathcal{S}$, and v is a "new" element that is not a vertex of any tree in $\mathcal{S}$, then T is in PayT where $\text{root}(T) = v$ and the edges of T are the edges of all the trees in $\mathcal{S}$ along with edges connecting $\text{root}(T)$ to the roots of each of the trees in $\mathcal{S}$. The trees in $\mathcal{S}$ are called the *children* of T.

We define functions max-value(T) and min-value(T) on payoff-game trees, $T \in$ PayT, recursively on the definition of PayT:

1. If T is a single node labelled r, then

$$\text{max-value}(T) = \text{min-value}(T) ::= r.$$

2. If the nonempty set, $\mathcal{S}$, is the set of children of T, then[1]

$$\begin{aligned}\text{max-value}(T) &::= \text{lub}\,\{\text{min-value}(S) \mid S \in \mathcal{S}\}\\ \text{min-value}(T) &::= \text{glb}\,\{\text{max-value}(S) \mid S \in \mathcal{S}\}\,.\end{aligned}$$

(a) Suppose a payoff-game tree, T, is *finite*. Prove that

[1]glb = "greatest-lower-bound" and lub = "least-upper-bound", *cf.*, Rosen, p. 423.

1. If the max-player is the first player to move in T, then he has a strategy that guarantees his payoff will be at least max-value(T), no matter how the min-player behaves.
2. If the max-player is the second player to move in T, then he has a strategy that guarantees his payoff will be at least min-value(T).
3. Likewise, if the min-player is the first player to move in T, then she has a strategy that guarantees the payoff to max-player will be at most min-value(T).
4. If the min-player is the second player to move in T, then she has a strategy that guarantees the payoff to max-player will be at most max-value(T).

(So the players may as well skip playing and just have the min-player pay max-value(T) to the max-player.)

(b) Now generalize the previous part to arbitrary PayT's. *Hint:* It might be helpful to assume the payoff amounts at the leaves are bounded above and below by particular numbers. After settling this case, try it without assuming bounds. Note that in the unbounded case, max-value(T) may be $+\infty$ and min-value(T) may be $-\infty$.

Problem 2. Week 6 Notes describes various functional equations, some of which serve as function definitions because there is only one function satisfying the equations, others of which are *ambiguous* because more than one function satisfies the equations, and some of which are *inconsistent* and aren't satisfied by any function at all. It can be hard to tell which case applies. In this problem, we'll consider some odd-ball equations which *do* turn out to define a function uniquely, though it's not obvious from the form of the equations that this is the case.

Suppose a function $f : \mathbb{N} \to \mathbb{N}$ satisfies the following condition:

$$f(n) = \begin{cases} n - 3 & \text{when } n \geq 10000 \\ f(f(n+5)) & \text{when } n < 10000. \end{cases} \tag{1}$$

(a) What the values of $f(n)$ be for n = 9996, 9997, 9998, 9999, and 10000?

(b) Prove that there is such a function, f, by giving a simple, explicit definition for f and proving that it satisfies the equations (1).

(c) Prove that equations (1) *uniquely* determine f. That is, if f is a function satisfying (1), then $f(n)$ equals the value you specified in part (b) for all $n \in \mathbb{N}$.

Problem 3. Closed Form Summations

Find a simplified closed form for each of the expressions listed below.

(a) $\sum_{i=x}^{y} 2i + 1$

(b) $\sum_{i=0}^{\infty} \sum_{j=1}^{n} \left(\frac{j}{j+2} \right)^{i}$

(c) $\prod_{i=1}^{n} 2 \cdot 4^{i}$.

Problem 4. Let $S_n = \sum_{i=1}^{n} i^{1/3}$. Use the integral method to determine a constant $c \in \mathbb{R}$ such that

$$S_n \sim c n^{\frac{4}{3}}.$$

Problem 5. Use the integral method to find upper and lower bounds for the following summation that differ by at most 0.1.

$$\sum_{i=1}^{\infty} \frac{1}{i^2}$$

Hint: Try adding the first few terms explicitly and then use integrals to bound the sum of the remaining terms.

(The actual value of the summation turns out to be $\pi^2/6 = 1.644\ldots.$)

Problem 6. Growing Choices

The expression $\binom{n}{k}$ is read "n choose k" and represents the quantity

$$\frac{n!}{k!\,(n-k)!}.$$

This expression comes up frequently in probability and combinatorics and will be used extensively later in this course. Suppose that n is even. Prove that

$$\binom{n}{n/2} \quad = \quad \Theta(2^n/\sqrt{n}).$$

Use the following form of Stirling's formula:

$$n! \quad \sim \quad \sqrt{2\pi n} \left(\frac{n}{e} \right)^n$$

Problem 7. Asymptotic Notation

Determine which of the eight choices below best describes each function's asymptotic behavior. Briefly indicate your reasoning; you may appeal to any of the results in the Notes or Rosen.

$\theta(n)$	$\theta(n!)$
$\theta(n \log n)$	$\theta(n^2)$
$\theta(1)$	$\theta(2^n)$
$2^{f(n)}$ where $f = \theta(n)$	None of the above

(a) $g(n) ::= n + \ln n + (\ln n)^2$.

(b) $h(n) ::= (n^2 + 2n - 3)/(n^2 - 1)$

(c) $j(n) ::= \sum_{i=0}^{n} 2^{2i+1}$

(d) $k(n) ::= (2 + \sin(n))2^{n+\sin(n)}$.

(e) $f(n) ::= \ln((n^2)!)$.