Solutions to In-Class Problems — Week 2, Mon

Problem 1. For each of the logical formulas, indicate whether or not it is true when the domain of discourse is $\mathbb{N}$ (the natural numbers 0, 1, 2, ...), $\mathbb{Z}$ (the integers), $\mathbb{Q}$ (the rationals), $\mathbb{R}$ (the real numbers), and $\mathbb{C}$ (the complex numbers).

$$\begin{array}{rcrcl} & \exists x & (x^2 & = & 2) \\ \forall x & \exists y & (x^2 & = & y) \\ \forall y & \exists x & (x^2 & = & y) \\ \forall x \neq 0 & \exists y & (xy & = & 1) \\ \exists x & \exists y & (x+2y & = & 2) \wedge (2x+4y=5) \end{array}$$

Solution.

Statement	$\mathbb{N}$	$\mathbb{Z}$	$\mathbb{Q}$	$\mathbb{R}$	$\mathbb{C}$
$\exists x\ (x^2 = 2)$	f	f	f	$t\ (x = \sqrt{2})$	t
$\forall x \exists y\ (x^2 = y)$	t	t	t	$t\ (y = x^2)$	t
$\forall y \exists x\ (x^2 = y)$	f	f	f	f (take $y < 0$)	t
$\forall x \neq 0 \exists y\ (xy = 1)$	f	f	t	$t\ (y = 1/x)$	t
$\exists x \exists y\ (x + 2y = 2) \wedge (2x + 4y = 5)$	f	f	f	f	f

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Problem 2. A media tycoon has an idea for an all-news television network called LNN: The Logic News Network. Each segment will begin with a definition of the domain of discourse and a few predicates. The day's happenings can then be communicated concisely in logic notation. For example, a broadcast might begin as follows:

> "THIS IS LNN. The domain of discourse is $\{\text{Bill}, \text{Monica}, \text{Ken}, \text{Linda}, \text{Betty}\}$. Let $D(x)$ be a predicate that is true if x is deceitful. Let $L(x, y)$ be a predicate that is true if x likes y. Let $G(x, y)$ be a predicate that is true if x gave gifts to y."

Complete the broadcast by translating the following statements into logic notation.

(a) If neither Monica nor Linda is deceitful, then Bill and Monica like each other.

Solution.

$$(\neg(D(\text{Monica}) \vee D(\text{Linda}))) \longrightarrow (L(\text{Bill}, \text{Monica}) \wedge L(\text{Monica}, \text{Bill}))$$

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(b) Everyone except for Ken likes Betty, and no one except Linda likes Ken.

Solution.

$$\forall x\ (x = \text{Ken} \wedge \neg L(x, \text{Betty})) \vee (x \neq \text{Ken} \wedge L(x, \text{Betty})) \wedge$$
$$\forall x\ (x = \text{Linda} \wedge L(x, \text{Ken})) \vee (x \neq \text{Linda} \wedge \neg L(x, \text{Ken}))$$

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(c) If Ken is not deceitful, then Bill gave gifts to Monica, and Monica gave gifts to someone.

Solution.

$$\neg D(\text{Ken}) \longrightarrow (G(\text{Bill}, \text{Monica}) \wedge \exists\, x G(\text{Monica}, x))$$

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(d) Everyone likes someone and dislikes someone else.

Solution.

$$\forall x \exists y \exists z\ (y \neq z) \wedge L(x, y) \wedge \neg L(x, z)$$

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(e) How could you express "Everyone except for Ken likes Betty" using just propositional connectives *without* using any quantifiers ($\forall, \exists$)? Can you generalize to explain how *any* logical formula over this domain of discourse can be expressed without quantifiers? How big would the formula in the previous part be if it was expressed this way?

Solution.

$$L(\text{Bill}, \text{Betty}) \wedge L(\text{Monica}, \text{Betty}) \wedge L(\text{Linda}, \text{Betty}) \wedge L(\text{Betty}, \text{Betty})$$

In general, quantifiers can be eliminated by treating $\forall x\ P(x)$ as an abbreviation for

$$P(\text{Bill}) \wedge P(\text{Monica}) \wedge P(\text{Ken}) \wedge P(\text{Linda}) \wedge P(\text{Betty}),$$

and $\exists x\ P(x)$ as an abbreviation for

$$P(\text{Bill}) \vee P(\text{Monica}) \vee P(\text{Ken}) \vee P(\text{Linda}) \vee P(\text{Betty}).$$

Expanded this way, the three-quantifier formula of the previous part would expand by a factor of $5 \times 5 \times 5 = 125$. So using quantifiers can pay off even when they are not strictly necessary. ■

Problem 3. Prove by induction that

$$\sum_{i=0}^{n} (2i+1)^2 = \frac{(n+1)(2n+1)(2n+3)}{3}$$

Solution. See Rosen, solution to §3.2, Exercise 9. ■