

## Homework 1

Due: February 9, 2005

**Note:** Recitation meets on Thursday, Feb 3 at 10am, 1pm and 4pm.

**Problem 1:** Let  $\mathbf{Z}$  be the set of Integers. Let  $R$  be the binary relation on  $\mathbf{Z}$  such that  $\{a, b\} \in R$  if and only if  $ab \geq 0$ .

- (a) Is  $R$  reflexive? Explain.
- (b) Is  $R$  symmetric? Explain.
- (c) Is  $R$  transitive? Explain.
- (d) Define a relation  $R_1 \subseteq R$  that is reflexive and symmetric but not transitive.
- (e) Define a relation  $R_2 \subseteq R$  that is reflexive and transitive but not symmetric.
- (f) Define a relation  $R_3 \subseteq R$  that is symmetric and transitive but not reflexive.
- (g) Define a relation  $R_4 \subseteq R$  that is an equivalence relation.

**Problem 2:** Construct truth tables for each of the following formulas. Also, for each pair of formulas, state which of the following holds:

- They are equivalent,
  - They are not equivalent, but one implies the other (say which is which), or
  - Neither of the above.
- (i)  $p \oplus (q \Rightarrow \neg p)$
  - (ii)  $(q \Rightarrow \neg p) \Rightarrow \neg p$
  - (iii)  $(p \Rightarrow q) \Rightarrow \neg p$
  - (iv)  $p \wedge \neg q \wedge (p \Rightarrow q)$

**Problem 3:**

- (a) Prove that for every natural number  $n \geq 12$ , there exist integers  $a, b$  such that  $n = 3a + 7b$ .
- (b) A *Hamiltonian cycle* in an undirected graph is a cycle that goes through every vertex in the graph exactly once.  
Suppose that  $G$  is an undirected graph that has a Hamiltonian cycle. Suppose that  $H$  is another undirected graph that is obtained from  $G$  by adding one node at a time,

along with some edges between the new node and some of the old nodes.  
More precisely, we have a sequence of graphs  $G = G_0, G_1, G_2, \dots, G_k = H$ , where each graph  $G_{i+1}$  is obtained from the previous graph  $G_i$  by adding one node  $n_{i+1}$ , together with edges connecting the new node  $n_{i+1}$  to strictly more than half of the nodes in the previous graph  $G_i$ . Prove that if  $G$  has a Hamiltonian cycle,  $H$  also must have a Hamiltonian cycle.