

Problem Set 4 Solutions

Solution to **Problem 1: Meet the Box People**

Solution to Problem 1, part a.

The probability that one of the box people's offspring has different phenotypes is as follows:

- i. An offspring has a circular phenotype unless s/he has genes cc. The probability of having cc is equal to the probability that each parent transmits gene c, squared (because they are independent events). Thus, this probability is $0.7^2 = 0.49$. This means that the probability that the offspring has a circular shape is $1 - 0.49 = 0.51$.
- ii. An offspring has a square phenotype when it inherits the recessive gene from both parents. Inheriting a recessive gene from one parent occurs with a probability of 0.7, thus the probability of inheriting a recessive gene from both parents is $0.7^2 = 0.49$, as before.
- iii. Using the same reasoning, we get that the probability of a blue phenotype is 0.25.
- iv. Similarly, the probability of a red phenotype is 0.75.

Solution to Problem 1, part b.

Given that an offspring is circular, the probability of other phenotypes are as follows:

- i. Because the genes are independent, the probability of having a blue phenotype given a circular shape is equal to the probability of having a blue phenotype (gene rr), which is 0.25.
- ii. Similarly, the probability of having a red phenotype given a circular shape is 0.75.

Solution to Problem 1, part c.

Given that an offspring is red, the probability of other phenotypes are as follows:

- i. Again, the genes are independent, thus the probability of a box shape given a red color is 0.49.
- ii. The probability of a circular shape given a red color is $1 - 0.49 = 0.51$.

Solution to Problem 1, part d.

We can draw a table of events as shown in Table 4-2.

We see that the probability that a person has the disease given that the test is positive, is:

$$\frac{0.001 \times 0.95}{0.001 \times 0.95 + 0.999 \times 0.004} = 19.2\% \quad (4-2)$$

Have Disease?	Percent	Test Results	Percent	Total
Yes	0.001	Positive	0.95	0.00095
		Negative	0.05	0.00005
No	0.999	Positive	0.004	0.003996
		Negative	0.996	0.95504

Table 4-2: Triangularity Test Results

Solution to Problem 2: Huffman Coding

Solution to Problem 2, part a.

To encode fourteen symbols we would need four bits, for $2^4 = 16$ different codewords. This gives $4 \times 44 = 176$ bits.

Solution to Problem 2, part b.

Table 4-3 lists the calculation of the average information per symbol. Here we calculate an average of 3.33 bits per symbol, or 147 bits.

Character	Frequency	$\log_2 \left(\frac{1}{p_i} \right)$	$p_i \log_2 \left(\frac{1}{p_i} \right)$
p	20.46%	2.29	0.46
e	18.18%	2.45	0.44
space	15.91%	2.65	0.42
c	6.82%	3.87	0.26
i	6.82%	3.87	0.26
k	6.82%	3.87	0.26
r	6.82%	3.87	0.26
d	4.55%	4.45	0.20
a	2.27%	5.46	0.12
f	2.27%	5.46	0.12
l	2.27%	5.46	0.12
o	2.27%	5.46	0.12
s	2.27%	5.46	0.12
t	2.27%	5.46	0.12
Total	100.00		3.33

Table 4-3: Frequency distribution of characters in “peter piper picked a peck of pickled peppers”

Solution to Problem 2, part c.

See Table 4-3.

Solution to Problem 2, part d.

A possible code is derived below and listed in Table 4-4.

Start: (p='NA' $p = 0.2046$) (e='NA' $p = 0.1818$) (space='NA' $p = 0.1591$) (c='NA' $p = .0682$) (i='NA' $p = .0682$) (k='NA' $p = .0682$) (r='NA' $p = .0682$) (d='NA' $p = .0455$) (a='NA' $p = .0227$) (f='NA' $p = .0227$) (l='NA' $p = .0227$) (o='NA' $p = .0227$) (s='NA' $p = .0227$) (t='NA' $p = .0227$)

Next: (p='NA' $p = 0.2046$) (e='NA' $p = 0.1818$) (space='NA' $p = 0.1591$) (c='NA' $p = .0682$) (i='NA' $p = .0682$) (k='NA' $p = .0682$) (r='NA' $p = .0682$) (d='NA' $p = .0455$) (a='NA' $p = .0227$) (f='NA' $p = .0227$) (l='NA' $p = .0227$) (o='NA' $p = .0227$) (s='0', t='1' $p = .0454$)

Next: (p='NA' $p = 0.2046$) (e='NA' $p = 0.1818$) (space='NA' $p = 0.1591$) (c='NA' $p = .0682$) (i='NA' $p = .0682$) (k='NA' $p = .0682$) (r='NA' $p = .0682$) (d='NA' $p = .0455$) (a='NA' $p = .0227$) (f='NA' $p = .0227$) (l='0', o='1' $p = .0454$) (s='0', t='1' $p = .0454$)

Next: (p='NA' $p = 0.2046$) (e='NA' $p = 0.1818$) (space='NA' $p = 0.1591$) (c='NA' $p = .0682$) (i='NA' $p = .0682$) (k='NA' $p = .0682$) (r='NA' $p = .0682$) (d='NA' $p = .0455$) (a='0', f='1' $p = .0454$) (l='0', o='1' $p = .0454$) (s='0', t='1' $p = .0454$)

Next: (p='NA' $p = 0.2046$) (e='NA' $p = 0.1818$) (space='NA' $p = 0.1591$) (c='NA' $p = .0682$) (i='NA' $p = .0682$) (k='NA' $p = .0682$) (r='NA' $p = .0682$) (d='NA' $p = .0455$) (a='0', f='1' $p = .0454$) (l='00', o='01', s='10', t='11' $p = .0908$)

Next: (p='NA' $p = 0.2046$) (e='NA' $p = 0.1818$) (space='NA' $p = 0.1591$) (c='NA' $p = .0682$) (i='NA' $p = .0682$) (k='NA' $p = .0682$) (r='NA' $p = .0682$) (d='0', a='10', f='11' $p = .0909$) (l='00', o='01', s='10', t='11' $p = .0908$)

Next: (p='NA' $p = 0.2046$) (e='NA' $p = 0.1818$) (space='NA' $p = 0.1591$) (c='NA' $p = .0682$) (i='NA' $p = .0682$) (k='0', r='1' $p = .1364$) (d='0', a='10', f='11' $p = .0909$) (l='00', o='01', s='10', t='11' $p = .0908$)

Next: (p='NA' $p = 0.2046$) (e='NA' $p = 0.1818$) (space='NA' $p = 0.1591$) (c='0', i='1' $p = .1364$) (k='0', r='1' $p = .1364$) (d='0', a='10', f='11' $p = .0909$) (l='00', o='01', s='10', t='11' $p = .0908$)

Next: (p='NA' $p = 0.2046$) (e='NA' $p = 0.1818$) (space='NA' $p = 0.1591$) (c='0', i='1' $p = .1364$) (k='0', r='1' $p = .1364$) (d='00', a='010', f='011', l='100', o='101', s='110', t='111' $p = .1817$)

Next: (p='NA' $p = 0.2046$) (e='NA' $p = 0.1818$) (space='NA' $p = 0.1591$) (c='00', i='01', k='10', r='11' $p = .2728$) (d='00', a='010', f='011', l='100', o='101', s='110', t='111' $p = .1817$)

Next: (p='NA' $p = 0.2046$) (e='NA' $p = 0.1818$) (c='00', i='01', k='10', r='11' $p = .2728$) (space='0', d='100', a='1010', f='1011', l='1100', o='1101', s='1110', t='1111' $p = .3408$)

Next: (p='0', e='1' $p = 0.3864$) (c='00', i='01', k='10', r='11' $p = .2728$) (space='0', d='100', a='1010', f='1011', l='1100', o='1101', s='1110', t='1111' $p = .3408$)

Next: (p='0', e='1' $p = 0.3864$) (c='000', i='001', k='010', r='011', space='10', d='1100', a='11010', f='11011', l='11100', o='11101', s='11110', t='11111' $p = .6136$)

Final: (p='00', e='01', c='1000', i='1001', k='1010', r='1011', space='110', d='11100', a='111010', f='111011', l='111100', o='111101', s='111110', t='111111' $p = 1.0000$)

Solution to Problem 2, part e.

When the sequence is encoded using the codebook derived in part d...

- i. See Table 4-5.
- ii. The fixed length code requires 176 bits, whereas Huffman coding requires 149 bits. So we find that the Huffman code does a better job than the fixed length code.

Character	Code
p	00
e	01
space	110
c	1000
i	1001
k	1010
r	1011
d	11100
a	111010
f	111011
l	111100
o	111101
s	111110
t	111111

Table 4-4: Huffman code for “peter piper picked a peck of pickled peppers”

Character	# of Characters	Bits per Character	Bits Needed
p	9	2	18
e	8	2	16
space	7	3	21
c	3	4	12
i	3	4	12
k	3	4	12
r	3	4	12
d	2	5	10
a	1	6	6
f	1	6	6
l	1	6	6
o	1	6	6
s	1	6	6
t	1	6	6
Total	44		149

Table 4-5: Huffman code for “peter piper picked a peck of pickled peppers”

- iii. This number compares extremely well with the information content of 147 bits for the message as a whole.

Solution to Problem 2, part f.

The original message is 44 bytes long, and with LZW we know from Problem Set 2 we can encode the message using LZW in 32 bytes, with 31 characters in the dictionary. Thus we need 32+14=46 different dictionary entries, for a total of six bits per byte. Thus we can compact the message down to $32 \times 6 = 192$ characters. Straight encoding needs 176 bits, and Huffman encoding needs 149 bits. Thus Huffman encoding does the best job of compacting the material.

A lower bound on sending the Huffman codebook is the number of bits in the code, total. This is equal to $2 + 2 + 3 + 4 + 4 + 4 + 4 + 4 + 5 + 6 + 6 + 6 + 6 + 6 + 6 = 64$ bits. If we imagine that we need to send some control bits along, perhaps it is something like five bits between each code (a reasonable estimate), this is an additional $5 \times (14 + 1) = 75$ bits. So we have an lower-bound estimate of 139 bits.

Thus a fixed-length code requires 176 bits, LZW needs 192 bits, and Huffman coding with the transmission of the codebook requires an estimated 296 bits.