Exercises

Exercise 22.1 Suppose you are given the single-input, nth-order system $x(k+1) = Ax(k) + bu(k)$, and assume the control u at every time step is confined to lie in the interval $[0, 1]$. Assume also that an eigenvalue of A, say λ_1, is real and nonnegative. Show that the set of states reachable from the origin is confined to one side of a hyperplane through the origin in $\mathcal{R}^n$. (Hint: An eigenvector associated with λ_1 will help you make the argument.)

[A hyperplane through the origin is an $(n-1)$-dimensional subspace defined as the set of vectors x in $\mathcal{R}^n$ for which $a'x = 0$, where a is some fixed nonzero vector in $\mathcal{R}^n$. Evidently a is normal to the hyperplane. The two "sides" of the hyperplane, or the two "half-spaces" defined by it, are the sets of x for which $a'x \leq 0$ and $a'x \geq 0$.]

Exercise 22.3 **(a)** Given m-input system $x(k+1) = Ax(k) + Bu(k)$, where A is the Jordan-form matrix

$$A = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}$$

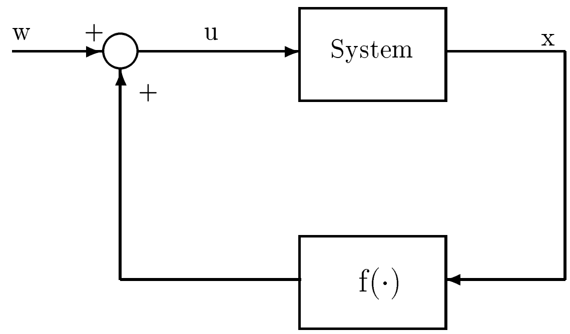
obtain conditions that are necessary and sufficent for the system to be reachable. (Hint: Your conditions should involve the *rows* b_i of B. Some form of the modal reachability test will — not surprisingly! — lead to the simplest solution.)

(b) Generalize this reachability result to the case where A is a general $n \times n$ Jordan-form matrix.

(c) Given the *single-input, reachable* system $x(k+1) = Ax(k) + bu(k)$, show that there can be only *one* Jordan block associated with each distinct eigenvalue of A.

Exercise 22.4 Given the n-dimensional reachable system $x(k+1) = Ax(k) + Bu(k)$, suppose that $u(k)$ is generated according to the nonlinear feedback scheme shown in the figure, where $u(k) = w(k) + f(x(k))$, with $f(.)$ being an arbitrary but known function, and $w(k)$ being the new control input for the closed-loop system.

Show that $w(k)$ can always be chosen to take the system state from the origin to any specified target state in no more than n steps. You will thereby have proved that *reachability is preserved under (even nonlinear) state feedback.*



$$x_{k+1} = Ax_k + B(w_k + f(x_k))$$

Exercise 23.1 Consider the single-input LTI system $\dot{x}(t) = Ax(t) + bu(t)$ with

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

We want to reach the target state $x_f = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ from the origin in 1 second.

(a) Find $e^{At}b$.

(b) Find the reachability Gramian G over an interval of length 1. Show that $x_f \in \mathcal{R}a(G)$, i.e. that $x_f = G\alpha$ for some α, and find α.

(c) Use your results from (a) and (b) to help you find an input $u(t)$ such that

$$\int_0^1 e^{A(1-t)} bu(t)\, dt = x_f$$

(i.e. an input that will take you from the origin at time 0 to the target state at time 1). Express the "energy" of this input, namely

$$\int_0^1 u^2(t) dt$$

in terms of G and α, and evaluate the result. How does this input compare with the minimum-energy input required to reach x_f from the origin in 1 second?

(d) If we choose a different target state, the energy of the input constructed by the above procedure will in general be different. Find a (possibly different) target state x_f with $\|x_f\|_2 = 1$ such that the energy of the input constructed by the above procedure is the *maximum* possible.

Exercise 23.4 Consider the perturbed Single-Input dynamic system:

$$\dot{x} = Ax + (b + \delta)u,$$

where $\delta \in \mathbb{R}^n$ is a perturbation vector. Assume that the nominal system (A, b) is reachable.

(a) Find the smallest $\|\delta\|_2$ so that the system is not reachable. This gives a robustness measure to the reachability of a system.

(b) To improve the robustness of reachability, an engineer suggested to apply a control input that consists of a feedback component; i.e.,

$$u = f^T x + v,$$

where $f \in \mathbb{R}^n$ and v is the external signal. She/He argued that for a special choice of f you need a larger δ (than part 1) to make the system not reachable. Do you agree with her/him? Prove or disprove this claim. (If you think it is true, it suffices to find one f that does the job. If you think it is not true, prove your claim).