

Lectures on Dynamic Systems and Control

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Chapter 12

Modal Decomposition of State-Space Models

12.1 Introduction

The solutions obtained in previous chapters, whether in time domain or transform domain, can be further decomposed to give a geometric understanding of the solution. The modal decomposition expresses the state equation as a linear combination of the various *modes* of the system and shows precisely how the initial conditions as well as the inputs impact these modes.

12.2 The Transfer Function Matrix

It is evident from (10.20) that the *transfer function matrix* for the system, which relates the input transform to the output transform when the initial condition is zero, is given by

$$H(z) = C(zI - A)^{-1}B + D. \quad (12.1)$$

For a multi-input, multi-output (MIMO) system with m inputs and p outputs, this results in a $p \times m$ matrix of rational functions of z . In order to get an idea of the nature of these rational functions, we express the matrix inverse as the adjoint matrix divided by the determinant, as follows:

$$H(z) = \frac{1}{\det(zI - A)} C [\text{adj}(zI - A)] B + D.$$

The determinant $\det(zI - A)$ in the denominator is an n^{th} -order monic (*i.e.* coefficient of z^n is 1) polynomial in z , known as the *characteristic polynomial* of A and denoted by $a(z)$. The

entries of the adjoint matrix (the cofactors) are computed from minors of $(zI - A)$, which are polynomials of degree less than n . Hence the entries of the matrices

$$(zI - A)^{-1} = \frac{1}{\det(zI - A)} \text{adj}(zI - A)$$

and

$$H(z) - D = \frac{1}{\det(zI - A)} C \text{adj}(zI - A) B$$

are strictly proper, *i.e.* have numerator degree strictly less than their denominator degree. With the D term added in, $H(z)$ becomes proper that is all entries have numerator degree less than or equal to the degree of the denominator. For $|z| \nearrow \infty$, $H(z) \rightarrow D$.

The polynomial $a(z)$ forms the denominators of all the entries of $(zI - A)^{-1}$ and $H(z)$, except that in some, or even all, of the entries there may be cancellations of common factors that occur between $a(z)$ and the respective numerators. We shall have a lot more to say later about these cancellations and their relation to the concepts of reachability (or controllability) and observability. To compute the inverse transform of $(zI - A)^{-1}$ (which is the sequence A^{k-1}) and the inverse transform of $H(z)$ (which is a matrix sequence whose components are the zero-state unit sample responses from each input to each output), we need to find the inverse transform of rationals whose denominator is $a(z)$ (apart from any cancellations). The roots of $a(z)$ — also termed the *characteristic roots* or *natural frequencies* of the system, thus play a critical role in determining the nature of the solution. A fuller picture will emerge as we proceed.

Multivariable Poles and Zeros

You are familiar with the definitions of poles, zeros, and their multiplicities for the scalar transfer functions associated with single-input, single-output (SISO) LTI systems. For the case of the $p \times m$ transfer function *matrix* $H(z)$ that describes the zero-state input/output behavior of an m -input, p -output LTI system, the definitions of poles and zeros are more subtle. We include some preliminary discussion here, but will leave further elaboration for later in the course.

It is clear what we would want our eventual definitions of MIMO poles and zeros to specialize to in the case where $H(z)$ is nonzero only in its *diagonal* positions, because this corresponds to completely decoupled scalar transfer functions. For this diagonal case, we would evidently like to say that the poles of $H(z)$ are the poles of the individual diagonal entries of $H(z)$, and similarly for the zeros. For example, given

$$H(z) = \text{diagonal} \left(\frac{z + 2}{(z + 0.5)^2}, \frac{z}{(z + 2)(z + 0.5)} \right)$$

we would say that $H(z)$ has poles of multiplicity 2 and 1 at $z = -0.5$, and a pole of multiplicity 1 at $z = -2$; and that it has zeros of multiplicity 1 at -2 , at $z = 0$, and at $z = \infty$. Note that

in the MIMO case we can have poles and zeros at the same frequency (e.g. those at -2 in the above example), without any cancellation! Also note that a pole or zero is not necessarily characterized by a single multiplicity; we may instead have a set of multiplicity indices (e.g. as needed to describe the pole at -0.5 in the above example). The diagonal case makes clear that we do *not* want to define a pole or zero location of $H(z)$ in the general case to be a frequency where *all* entries of $H(z)$ respectively have poles or zeros.

For a variety of reasons, the appropriate definition of a pole location is as follows:

- **Pole Location:** $H(z)$ has a pole at a frequency p_0 if *some* entry of $H(z)$ has a pole at $z = p_0$.

The full definition (which we will present later in the course) also shows us how to determine the set of multiplicities associated with each pole frequency. Similarly, it turns out that the appropriate definition of a zero location is as follows:

- **Zero Location:** $H(z)$ has a zero at a frequency η_0 if the *rank* of $H(z)$ *drops* at $z = \eta_0$.

Again, the full definition also permits us to determine the set of multiplicities associated with each zero frequency. The determination of whether or not the rank of $H(z)$ drops at some value of z is complicated by the fact that $H(z)$ may also have a pole at that value of z ; however, all of this can be sorted out very nicely.

12.3 Similarity Transformations

Suppose we have characterized a given dynamic system via a particular state-space representation, say with state variables x_1, x_2, \dots, x_n . The evolution of the system then corresponds to a trajectory of points in the state space, described by the succession of values taken by the state variables. In other words, the state variables may be seen as constituting the *coordinates* in terms of which we have chosen to describe the motion in the state space.

We are free, of course, to choose alternative coordinate bases — i.e., alternative state variables — to describe the evolution of the system. This evolution is not changed by the choice of coordinates; only the *description* of the evolution changes its form. For instance, in the LTI circuit example in the previous chapter, we could have used $i_L - v_C$ and $i_L + v_C$ instead of i_L and v_C . The information in one set is identical with that in the other, and the existence of a state-space description with one set implies the existence of a state-space description with the other, as we now show more concretely and more generally. The flexibility to choose an appropriate coordinate system can be very valuable, and we will find ourselves invoking such coordinate changes very often.

Given that we have a state vector x , suppose we define a constant invertible linear mapping from x to r , as follows:

$$r = T^{-1}x \quad , \quad x = Tr. \tag{12.2}$$

Since T is invertible, this maps each trajectory $x(k)$ to a unique trajectory $r(k)$, and vice versa. We refer to such a transformation as a *similarity transformation*. The matrix T embodies

the details of the transformation from x coordinates to r coordinates — it is easy to see from (12.2) that the columns of T are the representations of the standard unit vectors of r in the coordinate system of x , which is all that is needed to completely define the new coordinate system.

Substituting for $x(k)$ in the standard (LTI version of the) state-space model (10.1), we have

$$T r(k+1) = A (T r(k)) + B u(k) \quad (12.3)$$

$$y(k) = C (T r(k)) + D u(k). \quad (12.4)$$

or

$$r(k+1) = (T^{-1}AT) r(k) + (T^{-1}B) u(k) \quad (12.5)$$

$$= \hat{A} r(k) + \hat{B} u(k) \quad (12.6)$$

$$y(k) = (CT) r(k) + D u(k) \quad (12.7)$$

$$= \hat{C} r(k) + D u(k) \quad (12.8)$$

We now have a new representation of the system dynamics; it is said to be *similar* to the original representation. It is critical to understand, however, that the dynamic properties of the model are not at all affected by this coordinate change in the state space. In particular, the mapping from $u(k)$ to $y(k)$, *i.e.* the input/output map, is unchanged by a similarity transformation.

12.4 Solution in Modal Coordinates

The proper choice of a similarity transformation may yield a new system model that will be more suitable for analytical purposes. One such transformation brings the system to what are known as *modal coordinates*. We shall describe this transformation now for the case where the matrix A in the state-space model can be *diagonalized*, in a sense to be defined below; we leave the general case for later.

Modal coordinates are built around the *eigenvectors* of A . To get a sense for why the eigenvectors may be involved in obtaining a simple choice of coordinates for studying the dynamics of the system, let us examine the possibility of finding a solution of the form

$$x(k) = \lambda^k v, \quad v \neq 0 \quad (12.9)$$

for the undriven LTI system

$$x(k+1) = Ax(k) \quad (12.10)$$

Substituting (12.9) in (12.10), we find the requisite condition to be that

$$(\lambda I - A) v = 0 \quad (12.11)$$

i.e., that λ be an *eigenvalue* of A , and v an associated eigenvector. Note from (12.11) that multiplying any eigenvector by a nonzero scalar again yields an eigenvector, so eigenvectors are only defined up to a nonzero scaling; any convenient scaling or normalization can be used. In other words, (12.9) is a solution of the undriven system iff λ is one of the n roots λ_i of the *characteristic polynomial*

$$a(z) = \det(zI - A) = z^n + a_{n-1}z^{n-1} + \cdots + a_0 \quad (12.12)$$

and v is a corresponding eigenvector v_i . A solution of the form $x(k) = \lambda_i^k v_i$ is referred to as a *mode* of the system, in this case the i th mode. The corresponding λ_i is the i th *modal frequency* or *natural frequency*, and v_i is the corresponding *modal shape*. Note that we can excite just the i th mode by ensuring that the initial condition is $x(0) = \lambda_i^0 v_i = v_i$. The ensuing motion is then confined to the direction of v_i , with a scaling by λ_i at each step.

It can be shown fairly easily that eigenvectors associated with *distinct* eigenvalues are (linearly) *independent*, i.e. none of them can be written as a weighted linear combination of the remaining ones. Thus, *if* the n eigenvalues of A are distinct, then the n corresponding eigenvectors v_i are independent, and can actually form a *basis* for the state-space. Distinct eigenvalues are not necessary, however, to ensure that there exists a selection of n independent eigenvectors. In any case, we shall restrict ourselves for now to the case where — because of distinct eigenvalues or otherwise — the matrix A has n independent eigenvectors. Such an A is termed *diagonalizable* (for a reason that will become evident shortly), or *non-defective*. There do exist matrices that are *not* diagonalizable, as we shall see when we examine the Jordan form in detail later in this course.

Because (12.10) is linear, a weighted linear combination of modal solutions will satisfy it too, so

$$x(k) = \sum_{i=1}^n \alpha_i v_i \lambda_i^k \quad (12.13)$$

will be a solution of (12.10) for arbitrary weights α_i , with initial condition

$$x(0) = \sum_{i=1}^n \alpha_i v_i \quad (12.14)$$

Since the n eigenvectors v_i are independent under our assumption of diagonalizable A , the right side of (12.14) can be made equal to *any* desired $x(0)$ by proper choice of the coefficients α_i , and these coefficients are *unique*. Hence specifying the initial condition of the undriven system (12.10) specifies the α_i via (12.14) and thus, via (12.13), specifies the response of the undriven system. We refer to the expression in (12.13) as the *modal decomposition* of the undriven response. Note that the contribution to the modal decomposition from a conjugate pair of eigenvalues λ and λ^* will be a *real* term of the form $\alpha v \lambda^k + \alpha^* v^* \lambda^{*k}$.

From (12.14), it follows that $\alpha = V^{-1}x(0)$, where α is a vector with components α_i . Let $W = V^{-1}$, and w_i^t be the i^{th} row of W , then

$$x(k) = \sum_{i=1}^n \lambda_i^k v_i w_i^t x(0) \quad (12.15)$$

It is easy to see that w_i is a left eigenvector corresponding to the eigenvalue λ_i . The above modal decomposition of the undriven system is the same as obtaining the *diadic* form of A^k . The contribution of $x(0)$ to the i^{th} mode is captured in the term $w_i'x(0)$.

Before proceeding to examine the full response of a linear time-invariant model in modal terms, it is worth noting that the preceding results already allow us to obtain a precise condition for **asymptotic stability** of the system, at least in the case of diagonalizable A (it turns out that the condition below is the right one even for the general case). Recalling the definition in Example 10.1, we see immediately from the modal decomposition that the LTI system (12.10) is asymptotically stable iff $|\lambda_i| < 1$ for all $1 \leq i \leq n$, i.e. iff all the natural frequencies of the system are within the unit circle. Since it is certainly possible to have this condition hold even when $\|A\|$ is arbitrarily greater than 1, we see that the sufficient condition given in Example 1 is indeed rather weak, at least for the time-invariant case.

Let us turn now to the LTI version of the full system in (10.1). Rather than approaching its modal solution in the same style as was done for the undriven case, we shall (for a different point of view) approach it via a similarity transformation to modal coordinates, i.e., to coordinates defined by the eigenvectors $\{v_i\}$ of the system. Consider using the similarity transformation

$$x(k) = V r(k) \tag{12.16}$$

where the i th column of the $n \times n$ matrix V is the i th eigenvector, v_i :

$$V = \begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix} \tag{12.17}$$

We refer to V as the *modal matrix*. Under our assumption of diagonalizable A , the eigenvectors are independent, so V is guaranteed to be invertible, and (12.16) therefore does indeed constitute a similarity transformation. We refer to this similarity transformation as a *modal transformation*, and the variables $r_i(k)$ defined through (12.16) are termed *modal variables* or *modal coordinates*. What makes this transformation interesting and useful is the fact that the state evolution matrix A now transforms to a *diagonal* matrix Λ :

$$V^{-1}AV = \text{diagonal } \{\lambda_1, \dots, \lambda_n\} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = \Lambda \tag{12.18}$$

The easiest way to verify this is to establish the equivalent condition that $AV = V\Lambda$, which in turn is simply the equation (12.11), written for $i = 1, \dots, n$ and stacked up in matrix form. The reason for calling A “diagonalizable” when it has a full set of independent eigenvectors is now apparent.

Under this modal transformation, the *undriven* system is transformed into n *decoupled, scalar* equations:

$$r_i(k+1) = \lambda_i r_i(k) \tag{12.19}$$

for $i = 1, 2, \dots, n$. Each of these is trivial to solve: we have $r_i(k) = \lambda_i^k r_i(0)$. Combining this with (12.16) yields (12.13) again, but with the additional insight that

$$\alpha_i = r_i(0) \quad (12.20)$$

Applying the modal transformation (12.16) to the full system, it is easy to see that the transformed system takes the following form, which is once again decoupled into n parallel *scalar* subsystems:

$$r_i(k+1) = \lambda_i r_i(k) + \beta_i u(k), \quad i = 1, 2, \dots, n \quad (12.21)$$

$$y(k) = \xi_1 r_1(k) + \dots + \xi_n r_n(k) + Du(k) \quad (12.22)$$

where the β_i and ξ_i are defined via

$$V^{-1}B = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix}, \quad CV = \begin{bmatrix} \xi_1 & \xi_2 & \dots & \xi_n \end{bmatrix} \quad (12.23)$$

The scalar equations above can be solved explicitly by elementary methods (compare also with the expression in (22.2):

$$r_i(k) = \underbrace{\lambda_i^k r_i(0)}_{\text{ZIR}} + \underbrace{\sum_0^{k-1} \lambda_i^{k-\ell-1} \beta_i u(\ell)}_{\text{ZSR}} \quad (12.24)$$

where “ZIR” denotes the zero-input response, and “ZSR” the zero-state response. From the preceding expression, one can obtain an expression for $y(k)$. Also, substituting (12.24) in (12.16), we can derive a corresponding modal representation for the original state vector $x(k)$. We leave you to write out these details.

Finally, the same concepts hold for CT systems. We leave the details as an exercise.

Example 12.1

Consider the following system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 8 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u \quad (12.25)$$

We will consider the modal decomposition of this system for the zero input response. The eigenvalues of A are -4 and 2 and the associated eigenvectors are $\begin{bmatrix} 1 & -4 \end{bmatrix}'$ and $\begin{bmatrix} 1 & 2 \end{bmatrix}'$. The modal matrix is constructed from the eigenvectors above:

$$V = \begin{pmatrix} 1 & 1 \\ -4 & 2 \end{pmatrix} \quad (12.26)$$

Its inverse is given by

$$W = V^{-1} = \frac{1}{6} \begin{bmatrix} 2 & -1 \\ 4 & 1 \end{bmatrix}.$$

It follows that:

$$WAV = \Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} -4 & 0 \\ 0 & 2 \end{bmatrix}.$$

Now let's define r in modal coordinate as

$$x(t) = Tr \rightarrow r(t) = T^{-1}x(t).$$

Then in terms of r , the original system can be transformed into the following:

$$\begin{bmatrix} \dot{r}_1 \\ \dot{r}_2 \end{bmatrix} = \begin{bmatrix} -4 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}. \quad (12.27)$$

The response of the system for a given initial state and zero input can now be expressed as:

$$\begin{aligned} x(t) &= Vr(t) = Ve^{\Lambda(t-t_0)}Wx(t_0) \\ &= \begin{bmatrix} 1 & 1 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} e^{-4(t-t_0)} & 0 \\ 0 & e^{2(t-t_0)} \end{bmatrix} \frac{1}{6} \begin{bmatrix} 2 & -1 \\ 4 & 1 \end{bmatrix} x(t_0). \end{aligned}$$

For instance, if the initial vector is chosen in the direction of the first eigenvector, i.e., $x(t_0) = v_1 = [1 \quad -4]'$ then the response is given by:

$$x(t) = \begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{-4(t-t_0)}.$$

Example 12.2 Inverted Pendulum

Consider the linearized model of the inverted pendulum in Example 7.6 with the parameters given by: $m = 1$, $M = 10$, $l = 1$, and $g = 9.8$. The eigenvalues of the matrix A are 0, 0, 3.1424, and -3.1424 . In this case, the eigenvalue at 0 is repeated, and hence the matrix A may not be diagonalizable. However, we can still construct the Jordan form of A by finding the generalized eigenvectors corresponding to 0, and the eigenvectors corresponding to the other eigenvalues. The Jordan form of A , $\Lambda = T^{-1}AT$ and the corresponding transformation T are given by:

$$\Lambda = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3.1424 & 0 \\ 0 & 0 & 0 & -3.1424 \end{bmatrix}, T = \begin{bmatrix} 0.0909 & 0 & -0.0145 & 0.0145 \\ 0 & 0.0909 & -0.0455 & -0.0455 \\ 0 & 0 & 0.1591 & -0.1591 \\ 0 & 0 & 0.5000 & 0.5000 \end{bmatrix}$$

We can still get quite a bit of insight from this decomposition. Consider the zero input response, and let $x(0) = v_1 = [1 \ 0 \ 0 \ 0]'$. This is an eigenvector corresponding to the zero eigenvalue, and corresponds to a fixed distance s , zero velocity, zero angular position, and zero angular velocity. In that case, the system remains in the same position and the response is equal to $x(0)$ for all future time. Now, let $x(0) = v_2 = [0 \ 1 \ 0 \ 0]'$, which corresponds to a non-zero velocity and zero position, angle and angular velocity. This is not an eigenvector but rather a generalized eigenvector, i.e., it satisfies $Av_2 = v_1$. We can easily calculate the response to be $x(t) = [t \ 1 \ 0 \ 0]$ implying that the cart will drift with constant velocity but will remain in the upright position. Notice that the response lies in the linear span of v_1 and v_2 .

The case where $x(0) = v_3$ corresponds to the eigenvalue $\lambda = 3.1424$. In this case, the cart is moving to the left while the pendulum is tilted to the right with clockwise angular velocity. Thus, the pendulum tilts more to the right, which corresponds to unstable behavior. The case where $x(0) = v_4$ corresponds to the eigenvalue $\lambda = -3.1424$. The cart again is moving to the left with clockwise angular velocity, but the pendulum is tilted to the left. With an appropriate combination of these variables (given by the eigenvector v_4) the response of the system converges to the upright equilibrium position at the origin.

Exercises

Exercise 12.1 Use the expression in (12.1) to find the transfer functions of the DT versions of the controller canonical form and the observer canonical form defined in Chapter 8. Verify that the transfer functions are consistent with what you would compute from the input-output difference equation on which the canonical forms are based.

Exercise 12.2 Let v and w' be the right and left eigenvectors associated with some *non-repeated* eigenvalue λ of a matrix A , with the normalization $w'v = 1$. Suppose A is perturbed infinitesimally to $A + dA$, so that λ is perturbed to $\lambda + d\lambda$, v to $v + dv$, and w' to $w' + dw'$. Show that $d\lambda = w'(dA)v$.