MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science

6.241: Dynamic Systems—Fall 2003

Recitation 6

Lyapunov Methods

In this recitation we review concepts of stability and Lyapunov's direct and indirect methods for analysing the stability of a system around an equilibrium point. We then present a series of examples.

Definitions of Stability

Consider an unforced (time-invariant) nonlinear system described by $\dot{x}(t) = f(x(t))$. An *equilibrium point* of the system, $\bar{x}$, is one for which $f(\bar{x}) = 0$. Since any system with $\bar{x}$ not located at the origin can be translated to obtain an equivalent system with the equilibrium point at the origin (e.g., let $z = x - \bar{x}$), we assume in our definitions that the equilibrium point of our system is at the origin.

The system is called *stable in the sense of Lyapunov (i.s.L.)* around the equilibrium point at the origin if given any $\epsilon > 0$, there exists a $\delta > 0$ such that if $\|x(t_0)\| < \delta$ then $\|x(t)\| < \epsilon$ for all $t > t_0$. A system is called *asymptotically stable* around the equilibrium point at the origin if it is stable in the sense of Lyapunov and there exists a $\alpha > 0$ such that if $\|x(t_0)\| < \alpha$ then $x(t) \to 0$ as $t \to \infty$. If $lim_{t\to\infty} x(t) = 0$ no matter where $x(t_0)$ lies in the state space, then the system is said to be globally asymptotically stable.

Lyapunov's Direct Method

In general, proving that a nonlinear system of the form $\dot{x}(t) = f(x(t))$ is asymptotically stable around the origin is a difficult task, as it is difficult to write down a closed form solution of $x(t)$ in terms of $x(t_0)$. For linear time-invariant systems ($\dot{x}(t) = Ax(t) + Bu(t)$), we have a closed form solution, *i.e.*,

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau. \tag{1}$$

For any A (regardless of whether it is diagonalizable or not), the linear system $\dot{x} = Ax$ is asymptotically stable at the origin if and only if all the eigenvalues of A lie in the open left-half complex plane[1]. This is due to the decaying exponential terms in $x(t)$. On the other hand, if the system has at least one eigenvalue in the open right-half plane, then the linear system is unstable (i.e. not stable i.s.L.). Stability of the system (i.s.L.) if it has eigenvalues on the imaginary axis depends on the algebraic and geometric multiplicity of these eigenvalues (See lecture notes page 138).

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[1] We assume that the origin is the only equilibrium point, *i.e.*, A has full rank

To analyze the stability of a general nonlinear systems, we can carefully choose a scalar-valued function, $V(x)$, of the state variables and see how $V(x)$ evolves as the states evolve.

Lyapunov Function. Let V be a continuous map from $\mathbb{R}^n$ to $\mathbb{R}$, then $V(x)$ is called a *locally positive definite (lpd)* function around $x = 0$ if

1. $V(0) = 0$
2. $V(x) > 0$ on $0 < \|x\| < r_1 \quad$ for some r_1

V is called a Lyapunov function if V is differentiable, locally positive definite, and $\dot{V}$, the derivative of V along trajectories of the system, is locally negative semi-definite ($\dot{V}(x) \leq 0$ on $0 < \|x\| < r_2$ for some r_2).

Lyapunov Theorem for Local Stability: If there exists a Lyapunov function of the system $\dot{x} = f(x(t))$, then $x = 0$ is a stable equilibrium point i.s.L. If in addition, $\dot{V}(x) < 0$, for $0 < \|x\| < r$ for some r, then $x = 0$ is asymptotically stable (locally in the region $0 < \|x\| < r$).

Global asymptotic stability (G.A.S.) of the origin may be concluded if there exists a $V(x)$ that is positive definite and $\dot{V}(x)$ that is negative definite in the entire state space. In addition, V must be radially unbounded, that is, $V(x) \to \infty$ as $\|x\| \to \infty$. For a proof, see [1], page 109-111.

Lyapunov's Indirect Method

Instead of looking for a Lyapunov function to be applied directly to the nonlinear system, one can linearize the system around the origin and attempt to conclude "local" stability of the origin using quadratic Lyapunov functions for the linearized system. More specifically, if the linearized system's A matrix has eigenvalues in the open LHP, the nonlinear system is then locally asymptotically stable (L.A.S.). In addition, a Lyapunov function for the nonlinear system is quadratic, *i.e.,* $V(x) = x'Px$ where $P > 0$ solves the following Lyapunov equation: $A^T P + PA = -Q$, where $Q > 0$ is an arbitrary but positive definite matrix. See lecture 14 for details.

Examples

Example 1: In this example, we will prove global stability of the equilibrium point using the direct method.

$$\begin{aligned} \dot{x}_1 &= -x_1 - x_2 \\ \dot{x}_2 &= x_1 - x_2^3 \end{aligned}$$

First, note that the origin is the only equilibrium point of the system. Furthermore, linearizing about the origin, we obtain the dynamics of the linearized system,

$$\begin{bmatrix} \dot{\delta x}_1 \\ \dot{\delta x}_2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \delta x_1 \\ \delta x_2 \end{bmatrix}.$$

The eigenvalues of the linearization are given by $-0.5 \pm j0.5\sqrt{3}$, so we may conclude, via Lyapunov's indirect method, that the origin is locally asymptotically stable. However, the indirect method does not provide any information about how close to the origin we have to be to guarantee stability. Thus, we turn to the Lyapunov's direct method.

Consider using a candidate Lyapunov function $V(x)$ of the form $ax_1^2 + bx_2^2$ where a and b are some positive parameters to be determined. Clearly $V(x)$ is positive definite over the entire state space and $V(x)$ is radially unbounded. Now, we need to check that $\dot{V}(x)$ is negative definite:

$$\dot{V}(x) = \left[\begin{array}{cc} 2ax_1 & 2bx_2 \end{array}\right] \left[\begin{array}{c} -x_1 - x_2 \\ x_1 - x_2^3 \end{array}\right] = -2ax_1^2 - 2bx_2^4 + 2x_1x_2(b-a).$$

If we choose $a = b = 1/2$, then $\dot{V}(x) = -x_1^2 - x_2^4 < 0$ for all $x \neq 0$. Thus, this system is G.A.S.

Example 2: In this example, we find that the origin is locally asymptotically stable. We cannot conclude global asymptotic stability of the origin due to the presence of more than one equilibrium point; but, using Lyapunov's direct method, we can find the set of initial conditions for which the system trajectories will eventually end up at the origin (this ball is called the *region of attraction*).

Consider the autonomous (time-invariant) system described by the differential equations:

$$\begin{aligned} \dot{x}_1 &= (x_1 - x_2)(x_1^2 + x_2^2 - 1) \\ \dot{x}_2 &= (x_1 + x_2)(x_1^2 + x_2^2 - 1) \end{aligned}$$

This system has an infinite number of equilibrium points: one at the origin and the rest on the unit circle. Since there are multiple equilibria, none of the equilibria can be globally asymptotically stable; furthermore, since the points on the unit circle are not isolated, none can be locally asymptotically stable. We wish to examine the equilibrium point at the origin. Linearizing the system about the origin, we find that that the linearized dynamics have eigenvalues at $-1 \pm j$, so we can conclude, via the indirect method, local asymptotic stability (L.A.S.) of the origin.

Now, using the direct method, we can find the set of initial condtions for which the system trajectories converge to 0. Consider $V(x) = ax_1^2 + bx_2^2$, with $a, b > 0$, which is positive definite for all x_1 and x_2 (in $\mathbb{R}^2$). Now,

$$\dot{V}(x) = \left[\begin{array}{cc} 2ax_1 & 2bx_2 \end{array}\right] \left[\begin{array}{c} (x_1 - x_2)(x_1^2 + x_2^2 - 1) \\ (x_1 + x_2)(x_1^2 + x_2^2 - 1) \end{array}\right] = (2ax_1^2 + 2bx_2^2 + 2x_1x_2(b-a))(x_1^2 + x_2^2 - 1).$$

Note that $x_1^2 + x_2^2 - 1 < 0$ when $\|x\|_2 < 1$, $x \neq 0$. If $b = a = 1/2$, then $\dot{V}(x) = (x_1^2 + x_2^2)(x_1^2 + x_2^2 - 1) < 0$ for all nonzero x in the open unit circle. Thus the origin is L.A.S. within the region $\|x\|_2 < 1$.

Example 3: In this example, we see that sometimes it is hard to find a Lyapunov function. Still, we cannot always rush to the conclusion that the system is not stable, even if it is impossible to find a Lyapunov function for that system. For more on "converse" theorems (i.e. those that detail conditions for which a Lyapunov function is guaranteed to exist), see [1].

Analyze the stability of the origin for the system described by:

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_2^2 \\ \dot{x}_2 &= -x_2 \end{aligned}$$

This system has only one equilibrium point at the origin. Linearizing the system at the origin reveals that the system has two eigenvalues at -1, and the origin is therefore at least locally asymptotically stable. To make further conclusions about the stability of the origin (namely its region of attraction) we attempt to use a Lyapunov function of the form $V(x) = ax_1^2 + bx_2^2$, only to find that it is inconclusive. More specifically, in this case, $\dot{V}(x) = -2ax_1^2 + 2ax_1x_2^2 - 2bx_2^2$; this function is not negative definite for any nonzero values of a and b (note that we need $a, b > 0$ to keep $V(x)$ positive definite).

We might try to search for another Lyapunov function, but this is not an easy task. Alternatively, in this case, it is possible to find a closed form solution since x_2 evolves independently of x_1. Specifically,

$$x_2(t) = e^{-t}x_2(0)$$

so,

$$\dot{x}_1(t) = -x_1(t) + e^{-2t}x_2^2(0).$$

The e^{-2t} term can be seen as some input $u(t)$, and the closed form solution of the linear differential equation in x_1 can be obtained from equation 1 above,

$$x_1(t) = e^{-t}x_1(0) + \int_0^t e^{-(t-\tau)}x_2^2(0)e^{-2\tau}d\tau.$$

So,

$$\begin{aligned} x_1(t) &= x_1(0)e^{-t} + e^{-t}(1 - e^{-t})x_2^2(0) \\ x_2(t) &= x_2(0)e^{-t}. \end{aligned}$$

So, as $t \to \infty$, $x_1(t) \to 0$, and $x_2(t) \to 0$ for any initial conditions $x_2(0)$ and $x_1(0)$. Thus, this system is G.A.S.

Example 4: Finally, we have basically dealt with CT in this recitation, but Lyapunov theory extends to the DT case, $x(k+1) = f(x(k))$ with minor changes. Specifically, in DT, an equilibrium point, $\bar{x}$ is one for which $f(\bar{x}) = \bar{x}$, and we define $\dot{V}$ as follows, $\dot{V}(x) = V(f(x)) - V(x)$.

Consider the system (Example 13.7 of lecture notes):

$$\begin{aligned} x_1(k+1) &= \frac{x_2(k)}{1 + x_2^2(k)} \\ x_2(k+1) &= \frac{x_1(k)}{1 + x_2^2(k)}. \end{aligned}$$

This system has an equilibrium point at the origin. Now let $V(x) = x_1^2 + x_2^2$, which is positive definite, and evaluate $\dot{V}(x(k))$,

$$\dot{V}(x(k)) = V(x(k))\left(\frac{1}{[1 + x_2^2(k)]^2} - 1\right) \leq 0$$

Thus, we can conclude that the equilibrium point is stable i.s.L.

References

[1] Khalil, H.K. "Nonlinear Systems," *Prentice Hall*: 1996.