Exercises

Exercise 20.1 Consider a plant described by the transfer function matrix

$$P_\alpha(s) = \begin{pmatrix} \frac{\alpha}{s-1} & \frac{1}{s-1} \\ \frac{2s-1}{s(s-1)} & \frac{1}{s-1} \end{pmatrix}$$

where α is a real but uncertain parameter, confined to the range $[0.5\,,\,1.5]$. We wish to design a feedback compensator $K(s)$ for robust stability of a standard servo loop around the plant.

(a) We would like to find a value of α, say $\tilde{\alpha}$, and a scalar, stable, proper rational $W(s)$ such that the set of possible plants $P_\alpha(s)$ is contained within the "uncertainty set"

$$P_{\tilde{\alpha}}(s)[I + W(s)\Delta(s)]$$

where $\Delta(s)$ ranges over the set of stable, proper rational matrices with $\|\Delta\|_\infty \leq 1$. Try and find (no assurances that this is possible!) a suitable $\tilde{\alpha}$ and $W(s)$, perhaps by keeping in mind that what we really want to do is guarantee

$$\sigma_{max}\{P_{\tilde{\alpha}}^{-1}(j\omega)[P_\alpha(j\omega) - P_{\tilde{\alpha}}(j\omega)]\} \leq |W(j\omega)|$$

What specific choice of $\Delta(s)$ yields the plant $P_1(s)$ (i.e. the plant with $\alpha = 1$) ?

(b) Repeat part (a), but now working with the uncertainty set

$$P_{\tilde{\alpha}}(s)[I + W_1(s)\Delta(s)W_2(s)]$$

where $W_1(s)$ and $W_2(s)$ are column and row vectors respectively, and $\Delta(s)$ is scalar. Plot the upper bound on

$$\sigma_{max}\{P_{\tilde{\alpha}}^{-1}(j\omega)[P_\alpha(j\omega) - P_{\tilde{\alpha}}(j\omega)]\}$$

that you obtain in this case.

(c) For each of the cases above, write down a sufficient condition for robust stability of the closed-loop system, stated in terms of a norm condition involving the nominal complementary sensitivity function $T = (I + KP_{\tilde{\alpha}})^{-1}KP_{\tilde{\alpha}}$ and W — or, in part (b), W_1 and W_2.

Exercise 20.2 It turns out that the small gain theorem holds for nonlinear systems as well. Consider a feedback configuration with a stable system M in the forward loop and a stable, unknown perturbation in the feedback loop. Assume that the configuration is well-posed. Verify that the closed loop system is stable if $\|M\|\|\Delta\| < 1$. Here the norm is the gain of the system over *any* p-norm. (This result is also true for both DT and CT systems; the same proof holds).

Exercise 21.1 In decentralized control, the plant is assumed to be diagonal and controllers are designed independently for each diagonal element. If however, the real process is not completely decoupled, the interactions between these separate subsystems can drive the system to instability.

Consider the 2×2 plant

$$P(s) = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}.$$

Assume that P_{12} and P_{21} are stable and relatively small in comparison to the diagonal elements, and only a bound on their frequency response is available. Suppose a controller $K = diag(K_1, K_2)$ is designed to stabilize the system $P_0 = diag(P_{11}, P_{22})$.

1. Set-up the problem as a stability robustness problem, i.e., put the problem in the $M - \Delta$ form.
2. Derive a non-conservative condition (necessary and sufficient) that guarantees the stability robustness of the above system. Assume the off-diagonal elements are perturbed independently. Reduce the result to the simplest form (an answer like $\mu(M) < 1$ is not acceptable; this problem has an exact solution which is computable).
3. How does your answer change if the off-diagonal elements are perturbed simultaneously with the same Δ.

Exercise 21.2 Consider the rank 1 μ problem. Suppose $\mathbf{\Delta}$, contains only real perturbations. Compute the exact expression of $\mu(M)$.

Exercise 21.3 Consider the set of plants characterized by the following sets of numerators and denominators of the transfer function:

$$N(s) = N_0(s) + N_\delta(s)\delta, \qquad D(s) = D_0(s) + D_\delta(s)\delta$$

Where both N_0 and D_0 are polynomials in s, $\delta \in \mathbb{R}^n$, and N_δ, D_δ are polynomial row vectors. The set of all plants is then given by:

$$\Omega = \{\frac{N(s)}{D(s)} \,|\, \delta \in \mathbb{R}^n,\ |\delta_i| \leq \gamma\}$$

Let K be a controller that stabilizes $\frac{N_0}{D_0}$. Compute the exact stability margin; i.e., compute the largest γ such that the system is stable.