MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science

6.241: Dynamic Systems—Fall 2003

Homework 9 Solutions

Exercise 19.2 The characteristic polynom for the closed loop system is given by

$$s(s+2)(s+a)+1=0$$

Computing the locus of the closed poles as a function of a can be done numerically. The closed loop system is stable if $a \geq 0.225$. The above bound can also be derived by means of root locus techniques or by evaluating the Routh Hurwitz criterion. Another way of deriving bounds for the value of a is by casting this parametric uncertainty problem as an additive or multiplicative perturbation problem, see also 19.5. One can expect that the derived bounds in such a case would be rather conservative.

Exercise 19.3 The closed loop pole of the nominal plant is at $1-k$, so for nominal stability we need $k>1$. For robust stability let us apply small gain theorem for multiplicative perturbation:

$$\begin{aligned} \sup_{\omega}\left|\frac{4k}{(j\omega+10)(j\omega+K-1)}\right| &< 1, k>1 \\ \to \frac{k}{k-1} &< 2.5 \\ k &> \frac{5}{3} \end{aligned}$$

Note that in this case there is a limit on the gain from below, more like a "large gain theorem". We can also calculate characteristic polynomial of the perturbed system directly:

$$\chi(s)=s^2+(9+k)s+10k-10+4k\bar{\Delta}$$

where $\bar{\Delta}=\Delta/2$, i.e. uncertainty block normalized to unity. For stability of the second order polynomial both coefficients should be positive. Clearly the worst case is $\bar{\Delta}=-1$. In this case the requirement is $6k>10$, or $k>5/3$, the same result as obtained by using a small gain theorem.

Exercise 19.4 We can represent an uncertainty in feedback configuration, as shown below.

Note that the plant is SISO, and we consider blocks Δ and W to be SISO systems as well, so we can commute them. The transfer function seen by the Δ block can be derived as follows:

$$\begin{aligned} z &= P_0(-Ww-Kz) \\ &= -(I+P_0K)^{-1}P_0Ww \\ \therefore M &= -(I+P_0K)^{-1}P_0W. \end{aligned}$$

Apply the small gain theorem, and obtain the condition for stability robustness of the closed loop system as follows:

$$\sup_{\omega}\left|\frac{W(j\omega)P_0(j\omega)}{1+P_0(j\omega)K(j\omega)}\right|<1$$

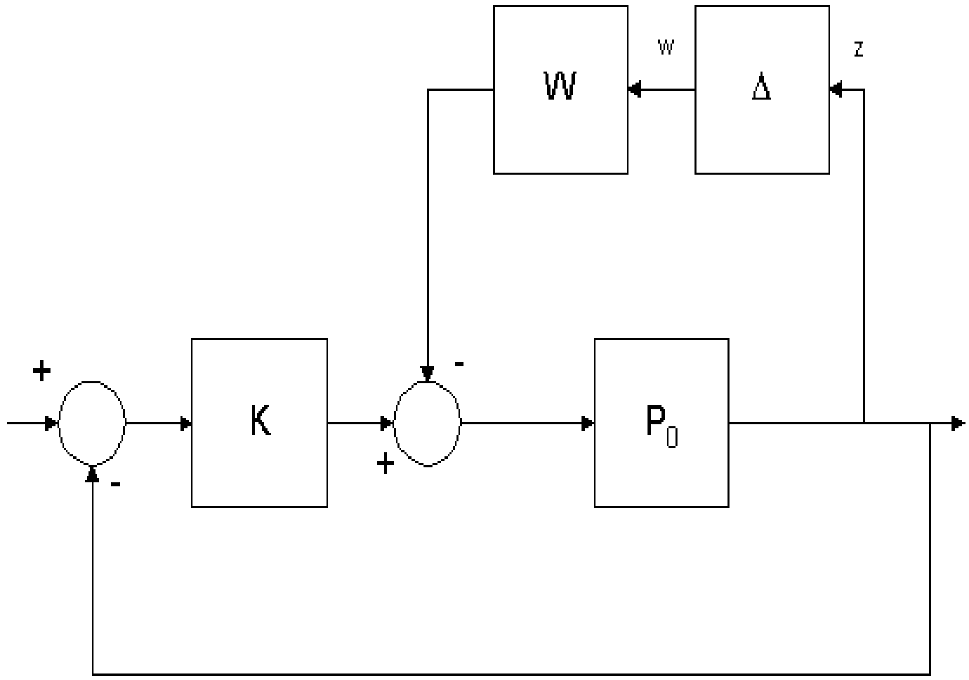

Figure 1: 19.4 Block Diagram.

Exercise 19.5 a) Given

$$P(s) = \frac{1}{s-a}\ ,\ K(s) = 10.$$

In order for the system to remain stable, the zeros of $1 + PK$ be in open left half plane. Thus,

$$1 + PK = 1 + \frac{10}{s-a} = \frac{s - s + 10}{s-a}\ \rightarrow\ a < 10.$$

b) Assume that the nominal plant is $P_0 = \frac{1}{s}$. With $W = -a$,

$$\Omega : \frac{P_0}{1 + W\Delta P_0} = \frac{\frac{1}{s}}{1 - a\Delta\frac{1}{s}} = \frac{1}{s - a\Delta},$$

so when $\Delta = 1$, we have

$$\frac{P_0}{1 + W\Delta P_0} = \frac{1}{s-a} = P,$$

which says that P is clearly in Ω.

c) The transfer function seen by the Δ block was derived is (from the previous problem):

$$M \quad = \quad -(I + P_0K)^{-1}P_0W.$$

Applying the small gain theorem:

$$\begin{aligned}
\|M\|_\infty &= \sup_\omega |(1+P_0K)^{-1}P_0K| < 1 \\
\rightarrow & \quad \sup_\omega |\frac{P_0W}{1+P_0K}| < 1 \\
\rightarrow & \quad \sup_\omega |\frac{\frac{-a}{j\omega}}{1+\frac{10}{j\omega}}| < 1 \\
& \vdots \\
|a| &< \sqrt{\omega^2+100} \\
\therefore |a| &< 10.
\end{aligned}$$

Since Δ block can have arbitrary phase we obtained much more conservative constraint on parameter a than the one in a).

d) Now, the nominal plant is replaced by $P_0 = \frac{1}{s+100}$. With the same description of Ω set, first in order to show that $P \in \Omega$ with $P_0 = \frac{1}{s+100}$, we need to find a new W as follows:

$$\frac{P_0}{1+W\Delta P_0} = \frac{\frac{1}{s+100}}{1+W\Delta\frac{1}{s+100}} = \frac{1}{s+100+W\Delta},$$

with $\Delta = 1$, the denominator becomes $s+100+W$, which we want to equate to $s-a$. Thus we have a new W to be

$$W = -a - 100.$$

Then in order to derive the condition on the closed loop system to be stable n the set Ω, we use the small gain theorem again.

$$\begin{aligned}
\|M\|_\infty &= \sup_\omega |(1+P_0K)^{-1}P_0K| < 1 \\
\rightarrow & \quad \sup_\omega |\frac{P_0W}{1+P_0K}| < 1 \\
\rightarrow & \quad \sup_\omega |\frac{\frac{-a-100}{j\omega+100}}{1+\frac{10}{j\omega+100}}| < 1 \\
|a+100| &< \sqrt{\omega^2+110^2} \\
& \vdots \\
\rightarrow -210 &< a < 10.
\end{aligned}$$

We can see that by representing uncertainty in a different way we can get a less conservative result.

Exercise 19.6 (a) Suppose we try to embed the set

$$\Pi = \left\{ P(z) = \frac{1}{z^{-1}-(1+b)} \qquad -2a_o \leq b \leq 2a_o \right\}$$

in a set of additive uncertainty; that is, every plant in the set Π is also in the set

$$\Omega_a = \{P(z) = P_o(z) + W(z)\Delta(z)\}$$

where W and Δ are stable and Δ has bounded norm. So, given any plant $P(z)$ in the set Π, we check that there is some stable Δ of bounded norm that will generate that $P(z)$ for some appropriately chosen (fixed) stable W. Now,

$$W\Delta = P - P_o = \frac{1}{z^{-1} - (1+b)} - \frac{1}{z^{-1} - (1+a_o)} = \frac{-a_o + b}{(z^{-1} - (1+b))(z^{-1} - (1+a_o))}$$

and note that if $b < 0$, $P - P_o$ has an unstable pole (outside the unit circle). Since there is no stable W and Δ that can give $P - P_o$ having an unstable pole, the additive uncertainty structure cannot cover all the elements in Π.

Now suppose we seek to embed Π in a set with multiplicative uncertainty perturbation Δ with bounded norm, i.e. $P = P_o(1 + W\Delta)$. Then,

$$W\Delta = \frac{P - P_o}{P_o} = \frac{-a_o + b}{(z^{-1} - (1+b))}.$$

Again, we cannot find a stable W and Δ that result in the unstable pole at $(b+1)^{-1}$ for $b < 0$.

(b) Suppose we embed Π in a set with uncertainty in feedback, i.e., $P = P_o(1 + W\Delta P_o)^{-1}$ (note, $P = P_o(1 + W\Delta)^{-1}$ also works). Then,

$$W\Delta = \frac{P - P_o}{P_o P} = -a_o + b,$$

now $-2a_o \leq b \leq 2a_o$ so choose $W = 3a_o$. Alternatively, if $P = P_o(1 + W\Delta)^{-1}$, we have

$$W\Delta = \frac{P - P_o}{P_o P} = \frac{-a_o + b}{z^{-1} - (1+a_o)},$$

so choose $W = \frac{3a_o}{z^{-1} - (1+a_o)}$. In either case, all the elements of Π will be generated by choosing an appropriate $\Delta = \delta$, where $\delta \in \mathbb{R}$ and $|\delta| \leq 1$.

For stability, we need $(I - M\Delta)^{-1}$ to be defined on and outside the unit circle, so we need that $(I - M\Delta)^{-1}$ has no poles on or outside the unit circle. So, using an argument similar to that used in the proof of the small gain theorem, we need to guarantee that $(I - M\Delta)^{-1}$ has no poles on the unit circle, i.e., $|\det((I - M\Delta)(e^{j\omega}))| > 0$, (where we have used the fact that $z = e^{j\omega}$ is the unit circle). For the plant having the structure $P = P_o(1 + W\Delta P_o)^{-1}$ and using the plant in the standard servo configuration, the transfer function from the output of the Δ block to the input of that block is $M = \frac{-P_o W}{1 + P_o K}$. So the stability robustness condition is

$$\left| 1 + \frac{P_o(e^{j\omega})W(e^{j\omega})}{1 + P_o(e^{j\omega})K(e^{j\omega})}\Delta(e^{j\omega}) \right| > 0.$$

Using a result shown at the end of chapter 19, it is sufficient to have

$$\left| \frac{P_o(e^{j\omega})W(e^{j\omega})}{1 + P_o(e^{j\omega})K(e^{j\omega})} \right| < 1 \quad \forall \omega.$$

(c) Let $T(e^{j\omega}) = \frac{P_o(e^{j\omega})W(e^{j\omega})}{1+P_o(e^{j\omega})K(e^{j\omega})}$, our stability robustness condition is $\left|1 + T(e^{j\omega})\Delta(e^{j\omega})\right| > 0$. As noted above, $\Delta = \delta$, where $\delta \in \mathbb{R}$ and $|\delta| \leq 1$, so $\left|1 + T(e^{j\omega})\delta\right| > 0$. In otherwords, we need $1 + T(e^{j\omega})\delta \neq 0$. Note that since δ is a real parameter, $1 + T(e^{j\omega})\delta = 0$ can only happen if $T(e^{j\omega})$ is real and equal to $\frac{-1}{\delta}$. So for all frequencies at which $T(e^{j\omega})$ is real-valued, we must have that $T(e^{j\omega}) \neq \frac{-1}{\delta}$. Since $|\delta| \leq 1$, we need that $|T(e^{j\omega})| < 1$ for all ω at which $T(e^{j\omega})$ is real valued, i.e. all ω for which the phase of $T(e^{j\omega})$ is 0 or π.