MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science

6.241: Dynamic Systems—Fall 2003

RECITATION 10

Stability Robustness with Structured Uncertainty

Robust stablilzation requires that the closed loop of the feedback configuration remain stable for any plant belonging to some uncertainty set. The small gain theorem gives a condition on the nominal closed loop system for which stability is guaranteed for all possible plants. The small gain theorm, however, is conservative (i.e., sufficient but not necessary) when the uncertainty in the system (namely, the Δ block in the M-Δ feedback configuration) has some structure.

Such structure may arise, for example, when there are several uncertainty blocks in the loop, as shown in the figure 1(b) below. Here, the disturbance rejection performance objective of minimizing the ratio of the energy of the output, y, to the energy of the disturbance, d, (see figure 1(a)) is made mathematically equivalent to the robust stabilization problem by introducing Δ_2. Recall that for disturbance rejection (assuming no uncertainty in the plant model) we required that $\|(I + P_oK)^{-1}W_2\|_\infty \leq \gamma$, for some γ, where $(I + P_oK)^{-1}W_2$ is the transfer function from d to y, and W_2 is a weighting matrix that captures information concerning the frequency distribution of the disturbance. Now, note that the transfer function from w_2 to z_2 in figure 1(b) is also $(I+P_oK)^{-1}W_2$. Thus, robust stabilization and the performance objective can be handled simultaneously by using a small gain-type argument on the system shown in figure 1(b).

To transform the system of figure 1(b) to the M-Δ form, we compute the transfer finction M from $w = \begin{bmatrix} w_1 & w_2 \end{bmatrix}^T$ to $z = \begin{bmatrix} z_1 & z_2 \end{bmatrix}^T$. It can be shown that in this case $\Delta = \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix}$ (see figure 2), so it has structure, hence, application of the small gain theorem gives a conservative condition for robust stability. To remedy this, the notion of structured singular value is introduced as a measure for robust stability.

Structured Singular Value

Let $M \in \mathbb{C}^{n\times n}$ the *structured singular value*, $\mu(M)$ is defined as follows:

$$\mu(M) = \frac{1}{\inf_{\Delta\in\tilde{\Delta}}\{\sigma_{max}(\Delta)|\det(I - M\Delta) = 0\}}.$$

If there is no $\Delta \in \tilde{\Delta}$ that makes $I - M\Delta$ singular, we define $\mu(M)$ to be zero.

Note that $\inf_{\Delta\in\tilde{\Delta}}\{\sigma_{max}(\Delta)|\det(I - M\Delta) = 0\}$ is the size, in the sense of σ_{max}, of the smallest Δ in the set $\tilde{\Delta}$ that makes $\det(I - M\Delta) = 0$.

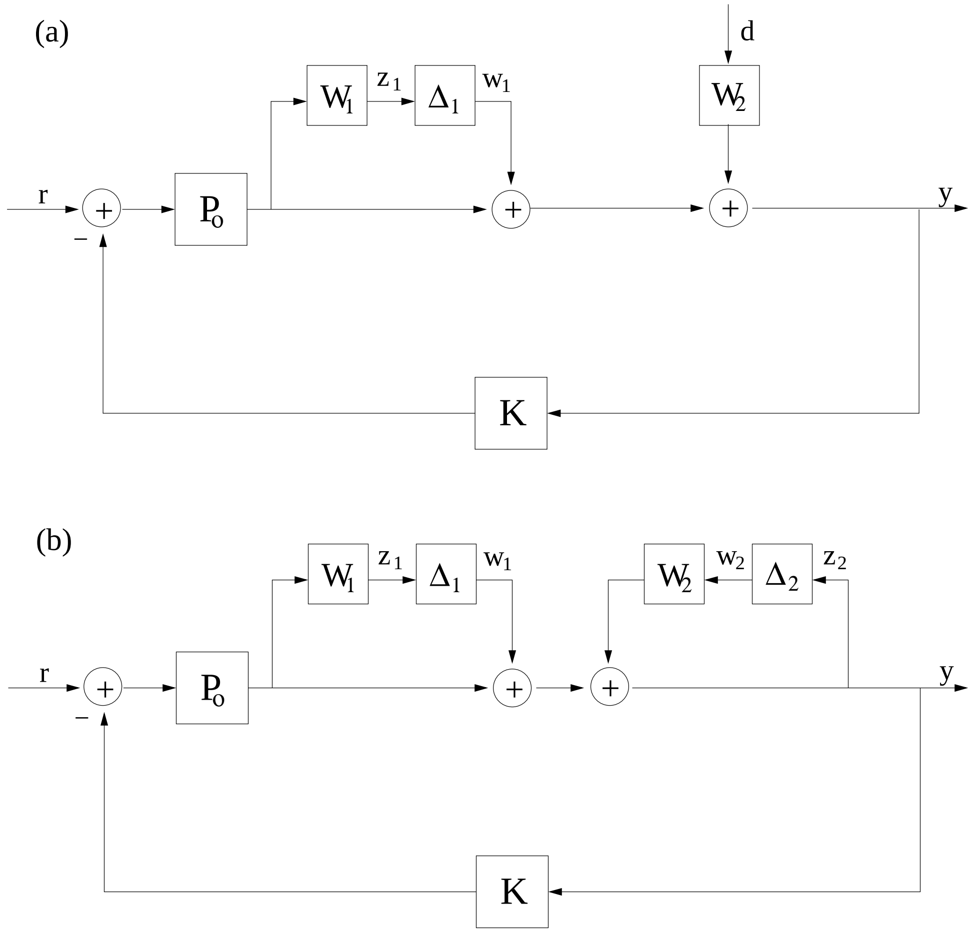

Figure 1: Robust Performance Model

In general, computing μ maybe a difficult task, however, we may be able to derive upper and lower bounds on μ. As we have previously seen, for $\tilde{\Delta}$ having a block-diagonal structure (i.e., all off-diagonal elements are zero), $\rho(M) \leq \mu(M) \leq \sigma_{max}(M)$. Still, the gap between ρ and σ_{max} may be large, so we seek to refine the bounds by transforming M in a way that does not affect $\mu(M)$ but does affect ρ and σ_{max}. In particular, the bounds maybe tightened to $\max_{U\in\mathcal{U}} \rho(UM) \leq \mu(M) \leq \inf_{D\in\mathcal{D}} \sigma_{max}(DMD^{-1})$ where, $\mathcal{U} = \{U \in \tilde{\Delta} | UU' = I\}$ and a matrix $D \in \mathcal{D}$ has zero off-diagonal elements, and its diagonal elements are, in general, $r_i \times r_i$ matrices, such that $D_i = D_i' > 0$.

Example 1:(From [[1] page 190]). Let $\Delta = \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{bmatrix}$. Suppose $M = \begin{bmatrix} 0 & \beta \\ 0 & 0 \end{bmatrix}$ where $\beta > 0$. We have $\rho(M) = 0$ and $\sigma_{max}(M) = \beta$, since $M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \beta & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Furthermore, $\mu(M) = 0$ because $\det(I - M\Delta) = 1$ for all admissible Δ :

$$I - M\Delta = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & \beta \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{bmatrix} = \begin{bmatrix} 1 & -\beta\delta_2 \\ 0 & 1 \end{bmatrix}.$$

So, the inequality $\rho(M) \leq \mu(M) \leq \sigma_{max}(M)$ does not give a good indication of what $\mu(M)$ actually is!

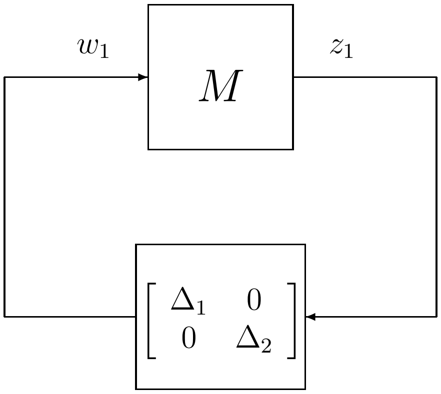

Figure 2: The Δ in the M-Δ configuration for the system of figure 1 has structure.

$M = \begin{bmatrix} -1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}$. In this case, $\rho(M) = 0$ and $\sigma_{max}(M) = 1$ and $\det(I - M\Delta) = 1 + \frac{\delta_1 - \delta_2}{2}$. So, $\min\{\max_i |\delta_i| | 1 + \frac{\delta_1 - \delta_2}{2} = 0\} = 1$ (see figure 3). So, $\mu(M) = 1$. Again, ρ and σ_{max} do not provide useful bounds.

Small Gain Theorem With Structured Uncertainty

Finally, we can state the robust stability condition in terms of $\mu(M)$. Let $\beta > 0$, the loop shown in figure 4 is well-posed and internally stable for all $\Delta \in \tilde{\Delta}$ with $\|\Delta\|_\infty < \frac{1}{\beta}$ if and only if $\sup_{\omega \in \mathbb{R}} \mu(M(j\omega)) \leq \beta$. The idea of the proof is as follows, when $\sup_{\omega \in \mathbb{R}} \mu(M(j\omega)) \leq \beta$, we have that the H_∞-norm of the smallest allowable destabilizing Δ, i.e., $\inf_{\Delta \in \tilde{\Delta}} \{\sigma_{max}(\Delta) | \det(I - M\Delta) = 0\} \geq \frac{1}{\beta}$. So all Δ with $\|\Delta\|_\infty < \frac{1}{\beta}$ cannot destabilize the system. On the other hand, suppose $\sup_{\omega \in \mathbb{R}} \mu(M(j\omega)) > \beta$, that is, there is some ω_o such that $\mu(M(j\omega_o)) > \beta$ then it can be shown that there exists a $\Delta \in \tilde{\Delta}$ with $\|\Delta\|_\infty < \frac{1}{\beta}$ such that $I - M(j\omega_o)\Delta(j\omega_o)$ is singular. This Δ can be constructed as was done in the proof of the small gain thorem, please refer to the lecture notes.

Rank One μ

In general, the structured singular value may be impossible to compute exactly, however, it is possible to compute $\mu(M)$ exactly when M has rank 1 and $\tilde{\Delta}$ is the set of block diagonal matrices. For example, M, the transfer function from $\begin{bmatrix} w_1 & w_2 \end{bmatrix}^T$ to $\begin{bmatrix} z_1 & z_2 \end{bmatrix}^T$ is rank one for the robust performance model shown in figure 1.

Since M has rank one, let $M = ab'$, where $a, b \in \mathbb{R}^n$ so we have that,

$$\begin{aligned} det(I - M\Delta) &= det(1 - ab'\Delta) \\ &= 1 - b'\Delta a \\ &= 1 - \begin{bmatrix} \Delta_1 & \ldots & \Delta_n \end{bmatrix} \begin{bmatrix} \bar{b}_1 a_1 \\ \vdots \\ \bar{b}_n a_n \end{bmatrix}. \end{aligned}$$

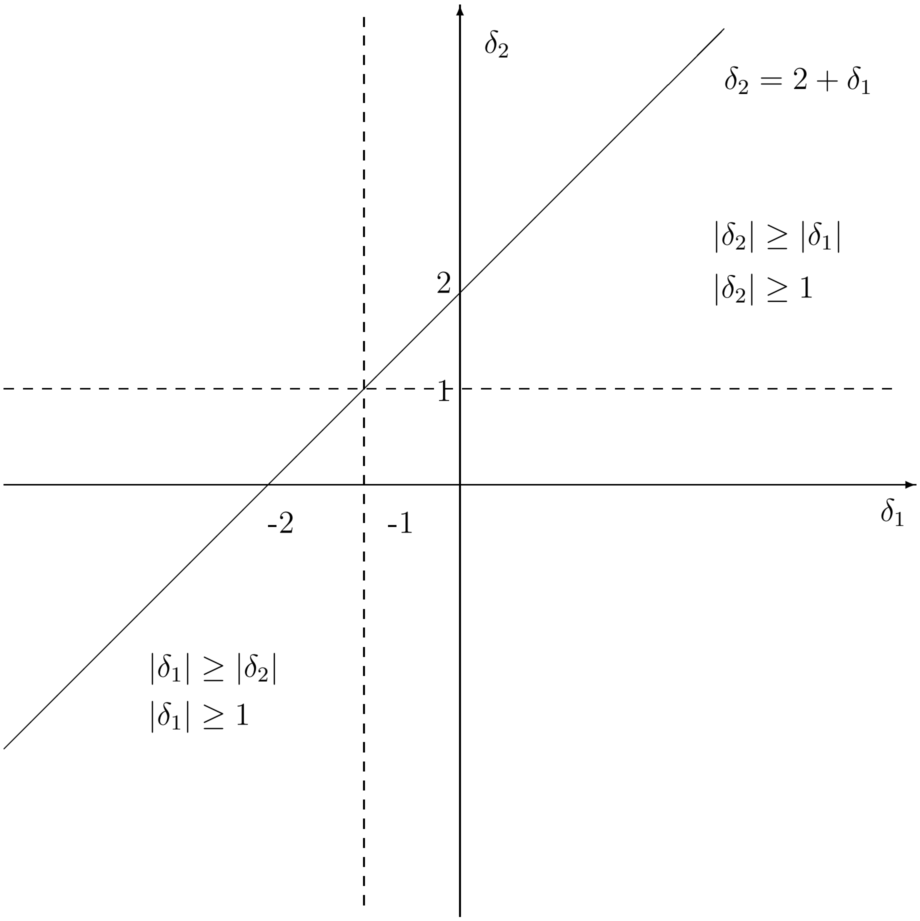

Figure 3: This figure shows that $\max\{|\delta_1|, |\delta_2|\} \geq 1$ for all (δ_1, δ_2) such that $1 + \frac{\delta_1 - \delta_2}{2} = 0$.

Assume that each of the Δ_i is single-input single-output, so $\sigma_{max}(\Delta) = \max_i |\Delta_i|$, so,

$$\frac{1}{\mu(M)} = \inf_{\Delta_1,\ldots,\Delta_n} \{\max_i |\Delta_i| | \begin{bmatrix} \Delta_1 & \ldots & \Delta_n \end{bmatrix} \begin{bmatrix} \bar{b}_1 a_1 \\ \vdots \\ \bar{b}_n a_n \end{bmatrix} = 1\}.$$

Now note that

$$1 = \sum_{i=1}^{n} \Delta_i \bar{b}_i a_i \leq \max_i |\Delta_i| \sum_{i=1}^{n} |\bar{b}_i a_i|.$$

Therefore,

$$\max_i |\Delta_i| \geq \frac{1}{\sum_{i=1}^{n} |\bar{b}_i a_i|}.$$

The lower bound can be achieved by setting $\Delta_i = \frac{sgn(\bar{b}_i a_i)}{\sum_{i=1}^{n} |b_i a_i|}$, where $sgn(x) = \frac{\bar{x}}{|x|}$ for $x \in \mathbb{C}$. Verify that with this choice, $|\Delta_i| = \frac{1}{\sum_{i=1}^{n} |\bar{b}_i a_i|}$ and $\sum_{i=1}^{n} \Delta_i \bar{b}_i a_i = 1$, as we require. Thus, if $\mu(M) = \sum_{i=1}^{n} |\bar{b}_i a_i| < 1$ for all w, then the smallest Δ that will result in a singular $I - M\Delta$ will have the $\mathcal{H}_\infty$-norm $\|\Delta\|_\infty = \frac{1}{\mu(M)} \geq 1$. So, as long as $\|\Delta\|_\infty \leq 1$, we are guaranteed that $\det(I - M\Delta) \neq 0$.

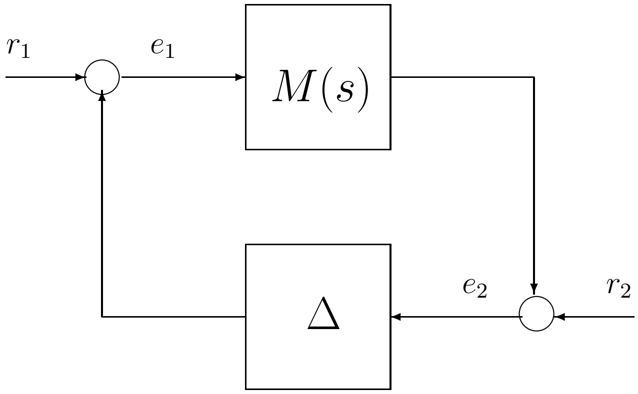

Figure 4: M-Δ Loop.

References

[1] Zhou, K., with Doyle, J. “Essentials of Robust Control.” *Prentice Hall*: 1998.