MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science

6.241: Dynamic Systems—Fall 2003

HOMEWORK 6 SOLUTIONS

Exercise 11.1.
Since the characteristic polynomial of A is a determinant of a matrix $zI - A$,

$$det(zI - A) = det((zI - A)^T) = det(zI - A^T),$$

first we show that

$$det(zI - A_1) = det(zI - A_2) = q(z)$$

for given A_1 and A_2. For

$$A_1 = \begin{pmatrix} -q_{n-1} & 1 & 0 & \cdots & 0 \\ -q_{n-2} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -q_1 & 0 & 0 & \cdots & 1 \\ -q_0 & 0 & 0 & \cdots & 0 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -q_0 & -q_1 & -q_2 & \cdots & -q_{n-1} \end{pmatrix},$$

we have

$$zI - A_1 = \begin{pmatrix} z + q_{n-1} & -1 & 0 & \cdots & 0 \\ q_{n-2} & z & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ q_1 & 0 & 0 & \cdots & -1 \\ q_0 & 0 & 0 & \cdots & z \end{pmatrix} \quad \text{and} \quad zI - A_2 = \begin{pmatrix} z & -1 & 0 & \cdots & 0 \\ 0 & z & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -1 \\ q_0 & q_1 & q_2 & \cdots & z + q_{n-1} \end{pmatrix}.$$

Recall that $det(A) = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in}$ for

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix},$$

where A_{ij} is a cofactor matrix corresponding a_{ij}. Then,

$$\begin{aligned} det(zI - A_1) \quad &= \quad (z+q_{n-1}) \begin{vmatrix} z & -1 & 0 & \cdots & 0 & 0 \\ 0 & z & -1 & \cdots & 0 & 0 \\ 0 & 0 & z & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & z & -1 \\ 0 & 0 & 0 & \cdots & 0 & z \end{vmatrix} - q_{n-2} \begin{vmatrix} -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & z & -1 & \cdots & 0 & 0 \\ 0 & 0 & z & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & z & -1 \\ 0 & 0 & 0 & \cdots & 0 & z \end{vmatrix} \\ &+ \quad q_{n-3} \begin{vmatrix} -1 & 0 & 0 & \cdots & 0 & 0 \\ z & -1 & 0\ldots & 0 & 0 & \\ 0 & 0 & z & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & z & -1 \\ 0 & 0 & 0 & \cdots & 0 & z \end{vmatrix} - \cdots \pm q_0 \begin{vmatrix} -1 & 0 & 0 & \cdots & 0 & 0 \\ z & -1 & 0 & \cdots & 0 & 0 \\ 0 & z & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 0 \\ 0 & 0 & 0 & \cdots & z & -1 \end{vmatrix}, \end{aligned}$$

where the last ± depends on whether n is an even or odd number. Similarly if we take the determinant of $zI - A_2$ using cofactors on the last row of $zI - A_2$ it is clear that we have

$$det(zI - A_1) = det(zI - A_2) = q(z).$$

Also it is true that

$$det(zI - A) = det((zI - A)^T) = det(zI - A^T).$$

Hence

$$det(zI - A_1) = det(zI - A_1^T) = det(zI - A_2) = det(zI - A_2^T) = q(z).$$

Then we have

$$\begin{aligned} q(z) \quad &= \quad (z + q_{n-1})z^{n-1} + q_{n-2}z^{n-2} + \cdots + q_1 z + q_0 \\ \therefore q(z) \quad &= \quad z^n + q_{n-1}z^{n-1} + q_{n-2}z^{n-2} + \cdots + q_1 z + q_0. \end{aligned}$$

b) For A_2, we have

$$\lambda_i I - A_2 = \begin{pmatrix} \lambda_i & -1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_i & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_i & -1 \\ q_0 & q_1 & q_3 & \cdots & q_{n-2} & \lambda_i + q_{n-1} \end{pmatrix}$$

Suppose v_1 is an eigenvector corresponding to λ_i, then $v_i \in \mathcal{N}(\lambda_i I - A_2)$, i.e.,

$$\begin{pmatrix} \lambda_i & -1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_i & -1 & \cdots & 0 & 0 \\ 0 & 0 & \lambda_i & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_i & -1 \\ q_0 & q_1 & q_3 & \cdots & q_{n-2} & \lambda_i + q_{n-1} \end{pmatrix} \underline{v}_i = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}. \qquad (1)$$

It is clear from Eqn 1 that the only nonzero $\underline{v}_i$ is

$$\underline{v}_i = \begin{pmatrix} 1 \\ \lambda_i \\ \lambda_i^2 \\ \vdots \\ \lambda_i^{n-1} \end{pmatrix}.$$

Then Eqn 1 becomes

$$\begin{pmatrix} \lambda_i \\ 0 \\ \vdots \\ q_0 \end{pmatrix} + \begin{pmatrix} -\lambda_i \\ \lambda_i^2 \\ \vdots \\ \lambda_i q_1 \end{pmatrix} + \cdots + \begin{pmatrix} 0 \\ \vdots \\ -\lambda_i^{n-1} \\ (\lambda_i + q_{n-1})\lambda_i^{n-1} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ q_0 + q_1\lambda_i + \cdots + \lambda_i^{n-1} q_{n-1} + \lambda_i^n \end{pmatrix} = \underline{0}$$

since λ_i is a root of $q(\lambda)$.

c) Consider

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & 5 & -2 \end{pmatrix}.$$

Its eigenvalues are $\lambda_1 = -1$, $\lambda_2 = -3$, and $\lambda_3 = 2$. Note that this A has the form of A_2 thus the corresponding eigenvectors can be written as follows:

$$\underline{v}_1 = \begin{pmatrix} 1 \\ \lambda_1 \\ \lambda_1^2 \end{pmatrix}, \ \underline{v}_2 = \begin{pmatrix} 1 \\ \lambda_2 \\ \lambda_2^2 \end{pmatrix}, \ \underline{v}_3 = \begin{pmatrix} 1 \\ \lambda_3 \\ \lambda_3^2 \end{pmatrix}.$$

Using those three eigenvectors we can obtain the similarity transformation matrix, M, to make A diagonal:

$$M = \begin{pmatrix} | & | & | \\ \underline{v}_1 & \underline{v}_2 & \underline{v}_3 \\ | & | & | \end{pmatrix}.$$

Thus with

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix},$$

we have

$$A = M\Lambda M^{-1},$$

which implies that

$$
\begin{aligned}
A^k &= M\Lambda^k M^{-1} = M \begin{pmatrix} \lambda_1^k & 0 & 0 \\ 0 & \lambda_2^k & 0 \\ 0 & 0 & \lambda_3^k \end{pmatrix} M^{-1} \\
&= \begin{pmatrix} 1 & 1 & 1 \\ -1 & -3 & 2 \\ 1 & 9 & 4 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & -\frac{1}{6} & -\frac{1}{6} \\ -\frac{1}{5} & -\frac{1}{10} & \frac{1}{10} \\ \frac{1}{5} & \frac{4}{v15} & \frac{1}{15} \end{pmatrix},
\end{aligned}
$$

and

$$
\begin{aligned}
e^{At} &= M \begin{pmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 \\ 0 & 0 & e^{\lambda_3 t} \end{pmatrix} M^{-1} \\
&= \begin{pmatrix} 1 & 1 & 1 \\ -1 & -3 & 2 \\ 1 & 9 & 4 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-3t} & 0 \\ 0 & 0 & e^{2t} \end{pmatrix} \begin{pmatrix} 1 & -\frac{1}{6} & -\frac{1}{6} \\ -\frac{1}{5} & -\frac{1}{10} & \frac{1}{10} \\ \frac{1}{5} & \frac{4}{15} & \frac{1}{15} \end{pmatrix}.
\end{aligned}
$$

Exercise 11.2. Given $\dot{x}(t) = Ax(t) + Bu(t)$ and $u(t) = u[k]$ for $kT < t \leq (k+1)T$.
a) Suppose $x[k] = x(kT)$, then

$$
\begin{aligned}
x[k+1] &= x((k+1)T) = e^{A(kT+T-t_0)}x(t_0) + \int_{t_0}^{kT+T} e^{A(kT+T-\tau)}Bu(\tau)d\tau \\
&= e^{AT}\{e^{A(kT-t_0)}x(t_0) + \int_{t_0}^{kT} e^{(kT-\tau)}Bu(\tau)d\tau + \int_{kT}^{kT+T} e^{A(kT-\tau)}Bd\tau u[k]\} \\
&= e^{AT}x[kT] + e^{AT}\int_{kT}^{kT+T} e^{A(kT-\tau)}Bd\tau u[k]
\end{aligned}
$$

Let $\alpha = \tau - kT$, then

$$
\begin{aligned}
x[k+1] &= e^{AT}x[k] + e^{AT}\int_0^T e^{-A\alpha}Bd\alpha u[k] \\
&= Fx[k] + Gu[k].
\end{aligned}
$$

b) Now we have

$$
A = \begin{pmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
$$

Let $F = e^{At}$, then using the inverse Laplace transform we have

$$
\begin{aligned}
(sI-A)^{-1} &= \begin{pmatrix} s & -1 \\ \omega_0^2 & s \end{pmatrix}^{-1} = \frac{1}{s^2+\omega_0^2}\begin{pmatrix} s & 1 \\ -\omega_0^2 & s \end{pmatrix} \\
\rightarrow e^{At} &= \begin{pmatrix} cos\omega_0 t & \frac{1}{\omega_0}sin\omega_0 t \\ -\omega_0 sin\omega_0 t & cos\omega_0 t \end{pmatrix} = F.
\end{aligned}
$$

Also we would like to compute the term with control input...

$$\begin{aligned} G &= \int_0^T \begin{pmatrix} cos\omega_0(T-u) & \frac{1}{\omega_0}sin\omega_0(T-u) \\ -\omega_0 sin\omega_0(T-u) & cos\omega_0(T-u)\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}du \\ &= \int_0^T \begin{pmatrix}\frac{1}{\omega_0}sin\omega_0(T-u)\\ cos\omega_0(T-u)\end{pmatrix}du \\ &= \begin{pmatrix}\frac{1}{\omega_0^2}(1-cos\omega_0T)\\ \frac{1}{\omega_0}sin\omega_0T\end{pmatrix}.\end{aligned}$$

c) To get the state $x[k]$ to reach to zero in at most two steps we need to have $x[2]=(F+GH)^2x[0]=0 \rightarrow (F+GH)^2=0$. Let's consider decomposing $F+GH$ into its Jordan form, i.e.,

$$F+GH = MJM^{-1}.$$

Then

$$(F+GH)^2 = (MJM^{-1})(MJM^{-1}) = MJ^2M^{-1}.$$

This implies that

$$(F+GH)^2 = 0 \leftrightarrow J^2 = 0 \rightarrow \lambda_1 = \lambda_2 = 0.$$

The trivial solution is $F+GH=0$, but we would like to find H such that $F+GH \neq 0$ but $(F+GH)^2=0$. With $H = \begin{pmatrix} H_1 & H_2 \end{pmatrix}$ and F and G as above, we have

$$\begin{aligned} F &= \begin{pmatrix} c & \frac{1}{\omega_0}s \\ -\omega_0 s & c \end{pmatrix},\ GH = \begin{pmatrix} \frac{1}{\omega_0^2}(1-c) \\ \frac{1}{\omega_0}s \end{pmatrix},\ \begin{pmatrix} H_1 & H_2 \end{pmatrix} = \begin{pmatrix} \frac{H_1}{\omega_0^2}(1-c) & \frac{H_2}{\omega_0^2}(1-c) \\ \frac{H_1}{\omega_0}s & \frac{H_2}{\omega_0}s \end{pmatrix} \\ \rightarrow F+GH &= \begin{pmatrix} c+\frac{H_1}{\omega_0^2}(1-c) & \frac{1}{\omega_0}s+\frac{H_2}{\omega_0^2}(1-c) \\ \frac{H_1}{\omega_0}s-\omega_0 s & \frac{H_2}{\omega_0}s+c \end{pmatrix}. \end{aligned}$$

Since the characteristic polynomial of $F+GH$ can be written as

$$q(\lambda) = \lambda^2 + trace(F+GH)\lambda + det(F+GH) = 0,$$

in order to have $\lambda_1=\lambda_2=0$ we would like to have $det(F+GH)=0$ and $trace(F+GH)=0$. For the determinant condition we have

$$\begin{aligned} det(F+GH) &= c^2 + \frac{H_2}{\omega_0}sc + \frac{H_1H_2s(1-c)}{\omega_0^3} + \frac{H_1}{\omega_0^2}(1-c)c \\ &- \left(\frac{H_1}{\omega_0^2}s^2 + \frac{H_1H_2}{\omega_0^3}s(1-c) - s^2 - \frac{H_2}{\omega_0}s(c1-c)\right) = 0. \end{aligned}$$

The trace condition gives

$$trace(F+GH) = 2c + \frac{H_2}{\omega_0}s + \frac{H_1}{\omega_0^2}(1-c) = 0.$$

Solving for H_1 and H_2, we get

$$\begin{aligned} H_1 &= \frac{\omega_0^2(2c-1)}{2(c-1)}, \\ H_2 &= -\frac{\omega_0(2c+1)}{2s}. \end{aligned}$$

Now, given $\omega_0 T = \frac{\pi}{6}$, $c = \frac{\sqrt{3}}{2}$, and $s = \frac{1}{2}$ so that we have

$$\begin{aligned} F &= \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2\omega_0} \\ -\frac{\omega_0}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \\ G &= \begin{pmatrix} \frac{1}{\omega_0^2}(1-\frac{\sqrt{3}}{2}) \\ \frac{1}{2\omega_0} \end{pmatrix} \\ H &= \begin{pmatrix} \omega_0^2\frac{(\sqrt{3}-1)}{\sqrt{3}-2} & -\omega_0(1+\sqrt{3}) \end{pmatrix} \\ \rightarrow F+GH &= \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}-2}{2\omega_0} \\ \frac{\omega_0}{2\sqrt{3}-4} & -\frac{1}{2} \end{pmatrix}. \end{aligned}$$

d) For $\omega_0 T = \frac{\pi}{6}$ and $\omega_0 = 1$ we have

$$F = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix},\ G = \begin{pmatrix} 1-\frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix},\ H = \begin{pmatrix} \frac{\sqrt{3}-1}{\sqrt{3}-2} & -(1+\sqrt{3}) \end{pmatrix}.$$

Then the open loop response and the closed response are shown in Fig.1 below:

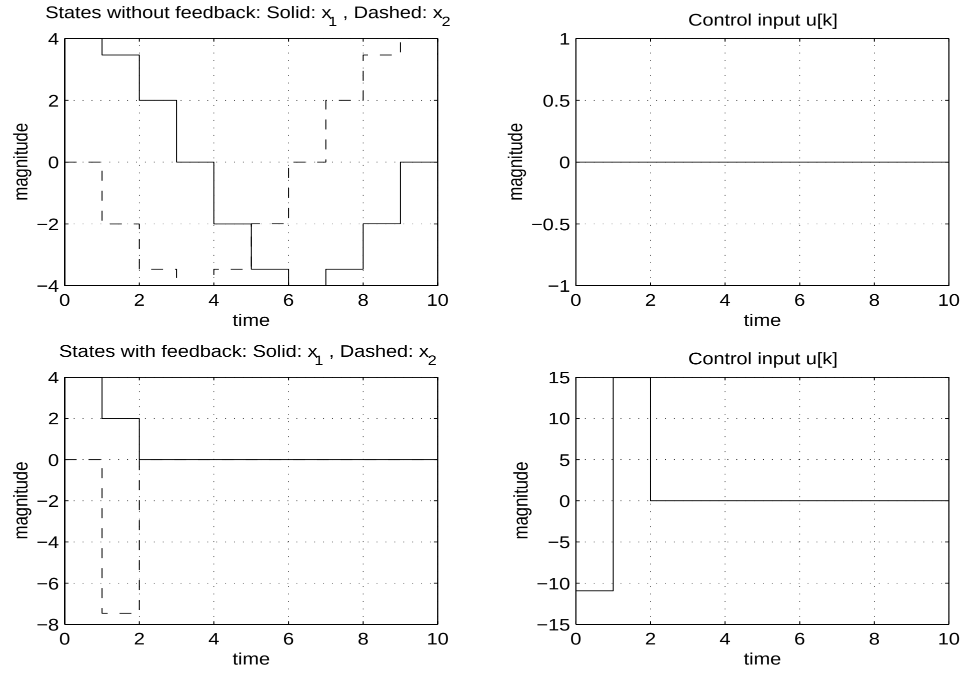

Figure 1: Open and closed loop responses of the system.

e) Now we have to consider the process delay to compute the control input so that we have the following system :

$$\begin{aligned} x[k+1] &= Fx[k] + Gu[k] \\ u[k] &= Hx[k-1]. \end{aligned}$$

Hence the difference equation form for the close loop system is written as :

$$x[k+1] = Fx[k] + GHx[k-1],$$

which can be written in state space matrix form as follows:

$$\begin{pmatrix} x[k+1] \\ x[k] \end{pmatrix} = \begin{pmatrix} F & GH \\ I & 0 \end{pmatrix} \begin{pmatrix} x[k] \\ x[k-1] \end{pmatrix}.$$

With the H computed in c), the eigenvalues for this system are $0.6325 \pm 1.3159i$, 0.4670, and 0. Thus this new closed system is unstable. The closed loop response for the delayed system is compared with the one without delay in Fig 2 below.

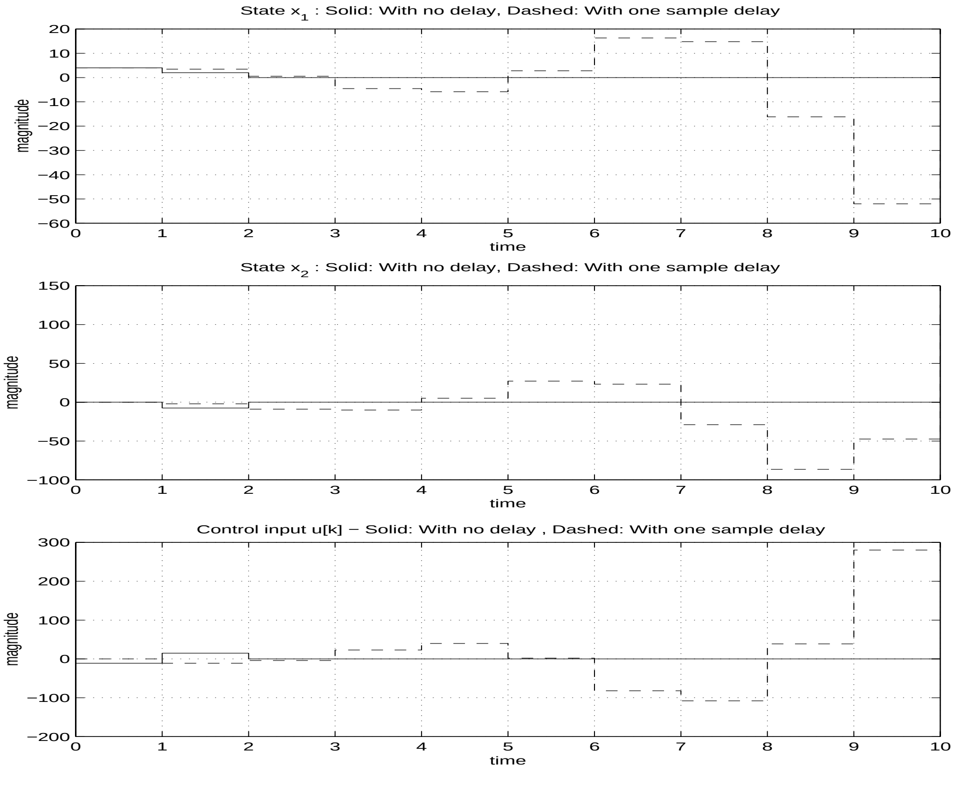

Figure 2: Closed loop response of the system with and without delay.

With this control law, we have a controllability matrix of rank 2, so that we can not control all the modes, i.e., not all the eigenvalues but two. One possible Modification to be made to have dead-beat behavior is to design an observer to estimate the all the four states, then use the same control structure used in the c), i.e., $u[k] = Hx[k]$ where now H is a 1×4 matrix. H should be chosen in such a way that the controllability matrix has the full rank, in this case 4. Then, since we can place the eigenvalues wherever we would like to, we can have deadbeat exactly for 2 steps. Yet, with this structure we need at least 2 steps since now the first step vector, $x[k+1]$ is included as a state vector.

Exercise 13.1 a) The system

$$\begin{aligned}\dot{x} &= z \\ \dot{z} &= -4x^3 + 2\alpha x = 4x\left(-x^2 + \frac{\alpha}{2}\right)\end{aligned}$$

has an equilibrium point at $(0, 0)$ for any value of α. Also, if $\alpha > 0$ there are two more equilibrium points: $\left(0, \pm\sqrt{\frac{\alpha}{2}}\right)$.

b) Linearizing the system around $(0, 0)$ we get the Jacobian:

$$A = \begin{pmatrix} 0 & 1 \\ 2\alpha & 0 \end{pmatrix}$$

The characteristic polynomial of the system is $det(A - \lambda I) = \lambda^2 - 2\alpha$. If $\alpha > 0$ there is an unstable root, if $\alpha < 0$ both roots are on imaginary axis, and the linearized system is neither asymtotically stable nor unstable (marginally stable). To analyze stability of the original non-linear system in this case we would have to look at the higher order terms. For the two other equilibrium points which exist for $\alpha > 0$ we get the Jacobian:

$$\begin{pmatrix} 0 & 1 \\ -4\alpha & 0 \end{pmatrix}$$

The characteristic polynomial for the system is $det(A - \lambda I) = \lambda^2 + 4\alpha$. Note $\alpha \leq 0$ does not concern us in this case since the equilibrium points $\left(0, \pm\sqrt{\alpha/2}\right)$ are valid when $0 < \alpha \leq 1$. If $\alpha > 0$ both roots lie on $j\omega$ axis and the system is marginally stable.

Exercise 13.2 a) Notice that the input-output differential equation can be written as

$$\ddot{y} = (\dot{u} - a_1\dot{y}) + (u - a_2 y - cy^2)$$

and we can use "observability-like" realization employed for a discrete-time system of exercise 7.1 (c). The differential equations for the states are

$$\begin{aligned}\dot{x_1} &= -a_1 x_1 + x_2 + u \\ \dot{x_2} &= -a_2 x_1 - c x_1^2 + u\end{aligned}$$

and the output equation is $y = x_1$. You can check that it is indeed a correct realization by differentiating the first state equation and plugging in an expression for $\dot{x_2}$ from the second equation.
b) Let us consider the system with zero input and $a_1 = 3$, $a_2 = 2$ and $c = 2$.

$$\begin{aligned}\dot{x_1} &= -3x_1 + x_2 \\ \dot{x_2} &= -2x_1 - 2x_1^2\end{aligned}$$

The linearized system's matrix is

$$A = \begin{pmatrix} -3 & 1 \\ -2 & 0 \end{pmatrix}$$

with the characteristic equation $\lambda^2 + 3\lambda + 2 = 0$, which has the roots $\lambda_1 = -1$ and $\lambda_2 = -2$. Therefore the linearized system is asymptotically stable around the origin, which also means that the original nonlinear system is a.s. around the origin (see lecture notes for the relevant theorem). You can also verify by linearization that the other equilibrium point $(-1, -3)$ is unstable.
c) Let us find a Lyapunov function for the linear system, and then find a region where its derivative is negative definite along the trajectories of the original nonlinear system. Since the linear system is asymptotically stable, for any symmetric positive definite matrix Q there exists a unique positive definite matrix P such that $A'P + PA = -Q$. Let us choose $Q = I$. Solving the system of linear equations imposed by the matrix relation we obtain

$$P = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix}$$

which gives us a quadratic Lyapunov function

$$V(x) = x'Px = \frac{1}{2}x_1^2 - x_1x_2 + x_2^2$$

Taking a derivative of $V(x)$ along the trajectory using the chain rule we get:

$$\dot{V}(x) = -x_1^2 - x_2^2 - 2x_1^2(2x_2 - x_1) = -x_1^2(1 + 2(2x_2 - x_1)) - x_2^2$$

The contour lines of the found Lyapunov function are

$$\frac{1}{2}x_1^2 - x_1x_2 + x_2^2 = C^2$$

for various constants C. Let us find such C that if the point $\begin{pmatrix} x_1 & x_2 \end{pmatrix}$ is within the boundary then $\dot{V}(x) < 0$. Then the trajectory started in this set will stay there, and will asymptotically decay to zero. Note that

$$V(x) = \frac{1}{4}x_1^2 + \frac{1}{4}(x_1 - 2x_2)^2 = C^2$$

therefore

$$|2x_2 - x_1| < 2C$$

Therefore if $C < 1/4$ the derivative is strictly less than zero. Hence we found a region of attraction as an ellipse, given by

$$x_1^2 + (x_1 - 2x_2)^2 < \frac{1}{4}$$

Any ball located completely within this ellipse will also be a region of attraction. Note also that this set is not exhaustive, there are other points in space that converge to zero.

Exercise 13.3 a) If $\bar{x}$ is a minimum of $P(x)$ then the gradient is equal to zero at that point $\frac{\partial P(\bar{x})}{\partial x} = 0$, therefore $\bar{x}$ is an equilibrium point of the system $\dot{x} = -\frac{\partial P}{\partial x}$.
b) Since $\bar{x}$ is a minimum $V(x)$ is positive definite: $P(x) - P(\bar{x}) > 0$. Since it is also an isolated minimum, there exists a ball surrounding $\bar{x}$ such that the gradient is non-zero. Therefore the derivative of $V(x)$ is strictly negative in that ball:

$$\dot{V}(x) \quad = \quad -\left\| \frac{\partial P}{\partial x} \right\|^2 < 0$$