

6.241: Dynamic Systems—Fall 2003

RECITATION 4

Convolutions, Laplace & \mathcal{Z} -Transforms

In this recitation, we review continuous-time and discrete-time convolution, as well as Laplace and z -transforms. You probably have seen these concepts in undergraduate courses, where you dealt mostly with one by one signals, $x(t)$ and $h(t)$. Concepts can be extended to cases where you have vectors of signals, for example, $x(t) = [x_1(t) \dots x_n(t)]'$, but now, one needs to be more careful with multiplication order. For example, as noted below, convolution is not commutative when vectors/matrices are involved, but are commutative when x and h are one by one.

Convolution

Continuous-Time Convolution. The convolution of x and h is defined by $h * x(t) = \int_{-\infty}^{+\infty} h(t - \tau)x(\tau)d\tau$.

It can be shown that the response of a linear system, with the impulse response¹ $h(t, \tau)$, to an input $x(t)$ is $y(t) = \int_{-\infty}^{+\infty} h(t, \tau)x(\tau)d\tau$ (see [1]). If the system is time-invariant, then the response of the system to $\delta(t - \tau)$, denoted by $h(t, \tau)$, is identical to a shifted version of its response to $\delta(t)$, i.e. $h(t - \tau, 0)$. Thus, if we let $h(t, 0) = h(t)$, then the response of an LTI system to any input $x(t)$ is given by the convolution integral.

Discrete-Time Convolution. The convolution of the discrete-time signals x and h is defined by $x * h[n] = \sum_{k=-\infty}^{+\infty} h[n - k]x[k]$.

It can be shown that the response of a linear system, with the impulse response² $h[n, k]$, to an input $x[n]$ is $y[n] = \sum_{k=-\infty}^{+\infty} h[n, k]x[k]$ (see [1]). If the system is time-invariant, then the response of the system to $\delta[n - k]$, denoted by $h[n, k]$, is identical to a shifted version of its response to $\delta[n]$, i.e. $h[n - k, 0]$. Thus, if we let $h[n, 0] = h[n]$, then the response of an LTI system to any input $x[n]$ is given by the convolution integral.

Example 1: Determine the response of a single input-single output continuous- (discrete-) time LTI system to the complex exponential input, e^{st} (z^n), where s (z) is a complex number. Let's plug into the convolution integral (sum). We have that $x(t) * h(t) = \int_{-\infty}^{+\infty} x(t - \tau)h(\tau)d\tau = \int_{-\infty}^{+\infty} e^{s(t-\tau)}h(\tau)d\tau = e^{st} \int_{-\infty}^{+\infty} e^{-s\tau}h(\tau)d\tau = e^{st}H(s)$ where, assuming $H(s)$ converges, $H(s) =$

¹Recall that the impulse response of the system is the output of the system, $h(t, \tau)$ when the input is the unit impulse function, $\delta(t - \tau)$.

²Recall that the impulse response of the system is the output of the system, $h[n, k]$ when the input is the unit impulse, $\delta[n - k]$.

$\int_{-\infty}^{+\infty} e^{-s\tau} h(\tau) d\tau$ is a complex constant whose value depends on the complex frequency, s . Similarly, $x[z] * h[z] = \sum_{k=-\infty}^{+\infty} x[n-k]h[k] = \sum_{k=-\infty}^{+\infty} z^{n-k}h[k] = z^n \sum_{k=-\infty}^{+\infty} z^{-k}h[k] = z^n H(z)$ where, assuming $H(z)$ converges, $H(z) = \sum_{k=-\infty}^{+\infty} z^{-k}h[k]$ is a complex constant whose value depends on the complex frequency, z .

Properties of LTI Systems. Since the response of an LTI system to any input can be obtained by convolving that input with the system's impulse response, LTI systems can be completely characterized by their impulse responses. Two other properties of convolutions/LTI systems are listed below:

- **Commutativity.** $x * h(t) = h * x(t)$ and $x * h[n] = h * x[n]$, *only when h and x are one by one* this can be directly verified by a substitution of variables in the convolution integral/sum. In general, if x or h were vectors or matrices, then convolution may not be commutative. A quick way to see this, consider $y = [y_1 \dots y_m]'$, and $x = [x_1 \dots x_n]'$, where $y[n] = \sum_{k=-\infty}^{+\infty} h[n-k]x[k]$. Here, h is an $m \times n$ matrix, and the order of multiplication of h and x cannot be flipped!
- **Distributivity.** $x(t) * [h_1(t) + h_2(t)] = x(t) * h_1(t) + x(t) * h_2(t)$ and $x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n]$, which can be directly verified from convolution integral/sum.
- **Causality for LTI systems.** The output of a causal system depends only on past or present values of the input. What, if anything, does this imply about the impulse response of an LTI system? Consider the single input-single output case (i.e. h and x are one by one). Going back to the convolution sum (or integral), $\sum_{k=-\infty}^{+\infty} h[n-k]x[k]$ ($\int_{-\infty}^{+\infty} h(t-\tau)x(\tau)d\tau$) note that, in general, the output at time n (or t), depends on past, present, and future values of the input. If the system were causal, then for any input, $x[n]$ (or $x(t)$), the output at time n (or t), must depend only on $x[k]$ for $k \leq n$ (or $x(t)$ or $\tau \leq t$). This happens if and only if $h[n-k] = 0$ for $k > n$ (or $h(t-\tau) = 0$ for $\tau > t$); equivalently, $h[n] = 0$ for $n < 0$ ($h(t) = 0$ for $t < 0$). Thus, for a causal LTI system, the convolution sum (integral) becomes: $\sum_{k=-\infty}^n h[n-k]x[k] = \sum_{k=0}^{+\infty} h[k]x[n-k]$ (or $\int_{-\infty}^t h(t-\tau)x(\tau)d\tau = \int_0^{+\infty} h(\tau)x(t-\tau)d\tau$.) Note that when the system is not single input single output, this condition on h is sufficient to ensure causality. And since future values of the input should not affect the present output, no matter what the input to the system is, this condition on h is also a necessary one.

Laplace and Z-Transforms

The Laplace transform (continuous time) and z-transform (discrete time) are important tools in the analysis of LTI systems. A set of differential (or difference) equations describing an LTI system in the time domain is conveniently reduced to a set of algebraic equations in the frequency domain; thus, the solution becomes easier to find in the frequency domain. Below, we present both the bilateral and unilateral Laplace and z-transforms.

Laplace Transform. The (bilateral) Laplace transform, $X(s)$ of a signal $x(t)$ is defined as $X(s) = \int_{-\infty}^{+\infty} e^{-st}x(t)dt$ for all s such that $X(s)$ is well-defined. The set of all such s is called the region of convergence (ROC).

Example 2: Find the Laplace transforms of the signals $x_1(t) = e^{-at}u(t)$ and $x_2(t) = -e^{-at}u(-t)$, where $a > 0$. We have $X_1(s) = \int_{-\infty}^{+\infty} e^{-at}u(t)e^{-st}dt = \int_0^{+\infty} e^{-(s+a)t}dt$, now let $s = \sigma + j\omega$, $X_1(\sigma + j\omega) = \int_0^{+\infty} e^{-(\sigma+a)t}e^{-j\omega t}dt$. Note that when $\sigma + a > 0$ i.e. $Re\{s\} > -a$, the integral is well defined and $X_1(s) = \frac{1}{s+a}$, $Re\{s\} > -a$. Similarly, $X_2(s) = -\int_{-\infty}^{+\infty} e^{-at}u(-t)e^{-st}dt = -\int_{-\infty}^0 e^{-(s+a)t}dt$, or $X_2(s) = \frac{1}{s+a}$, but now, for convergence we need that $Re\{s+a\} < 0$, i.e. $Re\{s\} < -a$.

Important properties of the Laplace transform are:

- Convolution. If $\mathcal{L}[x_1(t)] = X_1(s)$ with ROC= R_1 and $\mathcal{L}[x_2(t)] = X_2(s)$ with ROC= R_2 , then $\mathcal{L}[x_1(t) * x_2(t)] = X_1(s)X_2(s)$ with ROC containing $R_1 \cap R_2$.
- Differentiation. If $\mathcal{L}[x(t)] = X(s)$ with ROC= R , then $\mathcal{L}[\frac{d}{dt}x(t)] = sX(s)$ with ROC containing R .

The *unilateral* Laplace transform is important in analyzing causal systems, particularly when the system has nonzero initial conditions. The unilateral Laplace transform, $\mathcal{X}(s)$ of a signal $x(t)$ is defined as $\mathcal{X}(s) = \int_{0^-}^{+\infty} e^{-st}x(t)dt$ for all s such that $X(s)$ is well-defined. The set of all such s is called the region of convergence (ROC). Notice that the unilateral Laplace transform is the same as the bilateral transform when $x(t) = 0$ for all $t < 0$. The ROC of such a signal (hence the unilateral Laplace transform) is always the right-half plane. The convolution property of the unilateral Laplace transform is similar to that of the bilateral Laplace transform, namely, $\mathcal{UL}[x_1(t) * x_2(t)] = \mathcal{X}_1(s)\mathcal{X}_2(s)$ when both $x_1(t)$ and $x_2(t)$ are zero for $t < 0$. However, the differentiation property differs. Suppose $\mathcal{UL}[x(t)] = \mathcal{X}(s)$. Then, using integration by parts,

$$\begin{aligned} \mathcal{UL}[\frac{d}{dt}x(t)] &= \int_{0^-}^{+\infty} \frac{d}{dt}x(t)e^{-st}dt \\ &= [x(t)e^{-st}]_{0^-}^{\infty} + s \int_{0^-}^{+\infty} x(t)e^{-st}dt \\ &= s\mathcal{X}(s) - x(0^-) \end{aligned}$$

Z-Transform. The (bilateral) z-transform, $X(z)$ of a signal $x[n]$ is defined as $X(z) = \sum_{n=-\infty}^{+\infty} x[n]z^{-n}$ for all z such that $X(z)$ is well-defined. The set of all such z is called the region of convergence (ROC).

Example 3: Find the z-transforms of the signal $x[n] = a^n u[n]$. We have $X(z) = \sum_{n=-\infty}^{+\infty} a^n u[n]z^{-n} = \sum_{n=0}^{+\infty} (az^{-1})^n$. This converges when $|az^{-1}| < 1$, or equivalently, $|z| > |a|$.

Important properties of the z-transform are:

- Convolution. If $\mathcal{Z}\{x_1[n]\} = X_1(z)$ with ROC= R_1 and $\mathcal{Z}\{x_2[n]\} = X_2(z)$ with ROC= R_2 , then $\mathcal{Z}\{x_1[n] * x_2[n]\} = X_1(z)X_2(z)$ with ROC containing $R_1 \cap R_2$.
- Time Shifting. If $\mathcal{Z}\{x[n]\} = X(z)$ with ROC= R , then $\mathcal{Z}\{x[n - n_o]\} = z^{-n_o}X(z)$ with ROC= R , with possible addition or deletion of the origin or infinity.

The *unilateral* z-transform is important in analyzing causal systems, particularly when the system has nonzero initial conditions. The unilateral z-transform, $\mathcal{X}(z)$ of a signal $x[n]$ is defined as $\mathcal{X}[z] = \sum_{n=0}^{+\infty} x[n]z^{-n}$ for all z such that $X(z)$ is well-defined. The set of all such z is called the region of convergence (ROC). Notice that the unilateral z-transform is the same as the bilateral transform when $x[n] = 0$ for all $n < 0$. The ROC of such a signal (hence the unilateral z-transform) is the exterior of a circle. The convolution property of the unilateral z-transform is similar to that of the bilateral z-transform, namely, $\mathcal{U}\mathcal{Z}\{x_1[n] * x_2[n]\} = \mathcal{X}_1(z)\mathcal{X}_2(z)$ when both $x_1[n]$ and $x_2[n]$ are zero for $n < 0$. However, the time shifting property differs. Suppose $\mathcal{U}\mathcal{Z}\{x[n]\} = \mathcal{X}(z)$. Then, $\mathcal{U}\mathcal{Z}\{x[n+1]\} = z\mathcal{X}(z) - zx[0]$.

References

- [1] Oppenheim, A., and Willsky, A. "Signals and Systems," *Prentice Hall, Inc.*: 1997.
- [2] Kailath, T. "Linear Systems," *Prentice Hall, Inc.*: 1980.