Exercises

Exercise 1.1 Partitioned Matrices

Suppose

$$A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_4 \end{pmatrix}$$

with A_1 and A_4 *square*.

(a) Write the determinant $\det A$ in terms of $\det A_1$ and $\det A_4$. (Hint: Write A as the product

$$\begin{pmatrix} I & 0 \\ 0 & A_4 \end{pmatrix} \begin{pmatrix} A_1 & A_2 \\ 0 & I \end{pmatrix}$$

and use the fact that the determinant of the product of two *square* matrices is the product of the individual determinants — the individual determinants are easy to evaluate in this case.)

(b) Assume for this part that A_1 and A_4 are *nonsingular* (i.e., square and invertible). Now find A^{-1}. (Hint: Write $AB = I$ and partition B and I commensurably with the partitioning of A.)

Exercise 1.2 Partitioned Matrices

Suppose

$$A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$$

where the A_i are matrices of conformable dimension.

(a) What can A be premultiplied by to get the matrix

$$\begin{pmatrix} A_3 & A_4 \\ A_1 & A_2 \end{pmatrix} \quad ?$$

(b) Assume that A_1 is nonsingular. What can A be premultiplied by to get the matrix

$$\begin{pmatrix} A_1 & A_2 \\ 0 & C \end{pmatrix}$$

where $C = A_4 - A_3 A_1^{-1} A_2$?

(c) Suppose A is a square matrix. Use the result in (b) — and the fact mentioned in the hint to Problem 1(a) — to obtain an expression for $\det(A)$ in terms of determinants involving only the submatrices A_1, A_2, A_3, A_4.

Exercise 1.3 Matrix Identities

Prove the following *very useful* matrix identities. In proving identities such as these, see if you can obtain proofs that make as few assumptions as possible beyond those implied by the problem statement. For example, in (1) and (2) below, neither A nor B need be square, and in (3) neither B nor D need be square — so avoid assuming that any of these matrices is (square and) invertible!.

(a) $\det(I - AB) = \det(I - BA)$, if A is $p \times q$ and B is $q \times p$. (Hint: Evaluate the determinants of

$$\begin{pmatrix} I & A \\ B & I \end{pmatrix} \begin{pmatrix} I & -A \\ 0 & I \end{pmatrix}, \qquad \begin{pmatrix} I & -A \\ 0 & I \end{pmatrix} \begin{pmatrix} I & A \\ B & I \end{pmatrix}$$

to obtain the desired result). One common situation in which the above result is useful is when $p > q$; why is this so?

(b) Show that $(I - AB)^{-1}A = A(I - BA)^{-1}$.

(c) Show that $(A + BCD)^{-1} = A^{-1} - A^{-1}B(C^{-1} + DA^{-1}B)^{-1}DA^{-1}$. (Hint: Multiply the right side by $A + BCD$ and cleverly gather terms.) This is perhaps the most used of matrix identities, and is known by various names — the matrix inversion lemma, the $ABCD$ lemma (!), Woodbury's formula, etc. It is rediscovered from time to time in different guises. Its noteworthy feature is that, if A^{-1} is known, then the inverse of a modification of A is expressed as a modification of A^{-1} that may be simple to compute, e.g. when C is of small dimensions. Show, for instance, that evaluation of $(I - ab^T)^{-1}$, where a and b are column vectors, only requires inversion of a scalar quantity.

Exercise 1.4 Range and Rank

This is a practice problem in linear algebra (except that you have perhaps only seen such results stated for the case of real matrices and vectors, rather than *complex* ones — the extensions are routine).

Assume that $A \in \mathbf{C}^{m \times n}$ (i.e., A is a complex $m \times n$ matrix) and $B \in \mathbf{C}^{n \times p}$. We shall use the symbols $\mathcal{R}(A)$ and $\mathcal{N}(A)$ to respectively denote the range space and null space (or kernel) of the matrix A. Following the Matlab convention, we use the symbol A' to denote the transpose of the *complex conjugate of* the matrix A; $\mathcal{R}^{\perp}(A)$ denotes the subspace *orthogonal* to the subspace $\mathcal{R}(A)$, i.e. the set of vectors x such that $x'y = 0, \quad \forall y \in \mathcal{R}(A)$, etc.

(a) Show that $\mathcal{R}^{\perp}(A) = \mathcal{N}(A')$ and $\mathcal{N}^{\perp}(A) = \mathcal{R}(A')$.

(b) Show that

$$\operatorname{rank}(A) + \operatorname{rank}(B) - n \le \operatorname{rank}(AB) \le \min\{\operatorname{rank}(A), \operatorname{rank}(B)\}$$

This result is referred to as *Sylvester's inequality.*

Exercise 1.8 Let X be the vector space of polynomials of order less than or equal to M.

(a) Show that the set $B = \{1, x, \ldots x^M\}$ is a basis for this vector space.

(b) Consider the mapping T from X to X defined as:

$$f(x) = Tg(x) = \frac{d}{dx}g(x)$$

1. Show that T is linear.
2. Derive a matrix representation for T in terms of the basis B.
3. What are the eigenvalues of T.
4. Compute one eigenvector associated with one of the eighenvalues.