Exercises

Exercise 11.1 Companion Matrices

(a) The following two matrices and their transposes are said to be *companion matrices* of the polynomial $q(z) = z^n + q_{n-1}z^{n-1} + \ldots + q_0$. Determine the characteristic polynomials of these four matrices, and hence explain the origin of the name. (Hint: First find explanations for why all four matrices must have the same characteristic polynomial, then determine the characteristic polynomial of any one of them.)

$$A_1 = \begin{pmatrix} -q_{n-1} & 1 & 0 & \ldots & 0 \\ -q_{n-2} & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -q_1 & 0 & 0 & \ldots & 1 \\ -q_0 & 0 & 0 & \ldots & 0 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ -q_0 & -q_1 & -q_2 & \ldots & -q_{n-1} \end{pmatrix}$$

(b) Show that the matrix A_2 above has only one (right) eigenvector for each distinct eigenvalue λ_i, and that this eigenvector is of the form $[1 \ \ \lambda_i \ \ \lambda_i^2 \ \ \ldots \ \ \lambda_i^{n-1}]^T$.

(c) If

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & 5 & -2 \end{pmatrix}$$

what are A^k and e^{At}? (Your answers may be left as a product of three — or fewer — matrices; do not bother to multiply them out.)

Exercise 11.2 Suppose you are given the state-space equation

$$\dot{x}(t) = Ax(t) + Bu(t)$$

with an input $u(t)$ that is piecewise constant over intervals of length T:

$$u(t) = u[k] \ , \qquad kT < t \leq (k+1)T$$

(a) Show that the sampled state $x[k] = x(kT)$ is governed by a *sampled-data state-space model* of the form

$$x[k+1] = Fx[k] + Gu[k]$$

for constant matrices F and G (i.e. matrices that do not depend on t or k), and determine these matrices in terms of A and B. (Hint: The result will involve the matrix exponential, e^{At}.) How are the eigenvalues and eigenvectors of F related to those of A?

(b) Compute F and G in the above discrete-time sampled-data model when

$$A = \begin{pmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{pmatrix} , \qquad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

(c) Suppose we implement a *state feedback control law* of the form $u[k] = Hx[k]$, where H is a gain matrix. What choice of H will cause the state of the resulting closed-loop system, $x[k+1] = (F + GH)x[k]$, to go to 0 in at most two steps, from any initial condition (H is then said to produce "deadbeat" behavior)? To simplify the notation for your calculations, denote $\cos \omega_0 T$ by c and $\sin \omega_0 T$ by s. Assume now that $\omega_0 T = \pi/6$, and *check your result* by substituting in your computed H and seeing if it does what you intended.

(d) For $\omega_0 T = \pi/6$ and $\omega_0 = 1$, your matrices from (b) should work out to be

$$F = \begin{pmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{pmatrix} , \quad G = \begin{pmatrix} 1 - (\sqrt{3}/2) \\ 1/2 \end{pmatrix}$$

Use Matlab to compute and plot the response of each of the state variables from $k = 0$ to $k = 10$, assuming $x[0] = [4\,,\,0]^T$ and with the following choices for $u[k]$:

- (i) the open-loop system, with $u[k] = 0$;
- (ii) the closed-loop system with $u[k] = Hx[k]$, where H is the feedback gain you computed in (c), with $\omega_0 = 1$; also plot $u[k]$ in this case.

(e) Now suppose the controller is computer-based. The above control law $u[k] = Hx[k]$ is implementable if the time taken to compute $Hx[k]$ is negligible compared to T. Often, however, it takes a considerable fraction of the sampling interval to do this computation, so the control that is applied to the system at time k is forced to use the state measurement at the previous instant. Suppose therefore that $u[k] = Hx[k-1]$. Find a state-space model for the closed-loop system in this case, written in terms of F, G, and H. (Hint: The computer-based controller now has memory!) What are the eigenvalues of the closed-loop system now, with H as in (c)? Again use Matlab to plot the response of the system to the same initial condition as in (d), and compare with the results in (d)(ii). Is there another choice of H that could yield deadbeat behavior? If so, find it; if not, suggest how to modify the control law to obtain deadbeat behavior.

Exercise 13.1 Consider the horizontal motion of a particle of unit mass sliding under the influence of gravity on a frictionless wire. It can be shown that, if the wire is bent so that its height h is given by $h(x) = V_\alpha(x)$, then a state-space model for the motion is given by

$$\begin{aligned} \dot{x} &= z \\ \dot{z} &= -\frac{d}{dx} V_\alpha(x), \end{aligned}$$

Suppose $V_\alpha(x) = x^4 - \alpha x^2$.

(a) Verify that the above model has $(z,x) = (0,0)$ as equilibrium point for any α in the interval $-1 \leq \alpha \leq 1$, and it also has $(z,x) = \left(0, \pm\sqrt{\frac{\alpha}{2}}\right)$ as equilibrium points when α is in the interval $0 < \alpha \leq 1$.

(b) Verify that the linearized model about any of the equilibrium points is neither asymptotically stable nor unstable for any α in the interval $-1 \leq \alpha \leq 1$.

Exercise 13.2 Consider the dynamic system described below:

$$\ddot{y} + a_1\dot{y} + a_2 y + cy^2 = u + \dot{u},$$

where y is the output and u is the input.

(a) Obtain a state-space realization of dimension 2 that describes the above system.

(b) If $a_1 = 3$, $a_2 = 2$, $c = 2$, show that the system is asymptotically stable at the origin.

(c) Find a region (a disc of non-zero radius) around the origin such that every trajectory, with an initial state starting in this region, converges to zero as t approaches infinity. This is known as a region of attraction.

Exercise 13.3 Consider the system

$$\dot{x}(t) = -\frac{dP(x)}{dx}$$

where $P(x)$ has continuous first partial derivatives. The function $P(x)$ is referred to as the *potential function* of the system, and the system is said to be a *gradient* system. Let $\bar{x}$ be an isolated local minimum of $P(x)$, i.e. $P(\bar{x}) < P(x)$ for $0 < \|x - \bar{x}\| < r$, some r.

(a) Show that $\bar{x}$ is an equilibrium point of the gradient system.

(b) Use the candidate Lyapunov function

$$V(x) = P(x) - P(\bar{x})$$

to try and establish that $\bar{x}$ is an asymptotically stable equilibrium point.