Exercises

Exercise 5.1 Suppose the complex $m \times n$ matrix A is perturbed to the matrix $A + E$.

(a) Show that

$$|\,\sigma_{max}(A+E) \;-\; \sigma_{max}(A)\,| \;\leq\; \sigma_{max}(E)$$

Also find an E that results in the inequality being achieved with equality.

(Hint: To show the inequality, write $(A+E) = A + E$ and $A = (A+E) - E$, take the 2-norm on both sides of each equation, and use the triangle inequality.)

It turns out that the result in (a) actually applies to *all* the singular values of A and $A+E$, not just the largest one. Part (b) below is one version of the result for the smallest singular value.

(b) Suppose A has *less than* full column rank, i.e. has rank$< n$, but $A + E$ has full column rank. Show (following a procedure similar to part (a) — but looking at $\min \|(A+E)x\|_2$ rather than the norm of $A + E$, etc.) that

$$\sigma_{min}(A+E) \;\leq\; \sigma_{max}(E)$$

Again find an E that results in the inequality being achieved with equality.

[The result in (b), and some extensions of it, give rise to the following sound (and widely used) procedure for estimating the rank of some underlying matrix A, given only the matrix $A + E$ and knowledge of $\|E\|_2$: Compute the SVD of $A+E$, then declare the "numerical rank" of A to be the number of singular values of $A + E$ that are larger than the threshold $\|E\|_2$. The given information is consistent with having an A of this rank.]

(c) Verify the above results using your own examples in MATLAB. You might also find it interesting to verify numerically that for large m, n, the norm of the matrix $E = s * randn(m,n)$ — which is a matrix whose entries are independent, zero-mean, Gaussian, with standard deviation s — is close to $s * (\sqrt{m} + \sqrt{n})$. So if A is perturbed by such a matrix, then a reasonable value to use as a threshold when determining the numerical rank of A is this number.

Exercise 5.2 Let A and E be $m \times n$ matrices. Show that

$$\min_{\text{rank } E \leq r} \|A - E\|_2 = \sigma_{r+1}(A).$$

To prove this, notice that the rank constraint on E can be interpreted as follows: If $v_1, \ldots, v_{r+1}$ are linearly independent vectors, then there exists a nonzero vector z, expressed as a linear combination of such vectors, that belongs to the nullspace of E. Proceed as follows:

1. Select the v_i's from the SVD of A.
2. Select a candidate element z with $\|z\|_2 = 1$.
3. Show that $\|(A-E)z\|_2 \geq \sigma_{r+1}$. This implies that $\|A - E\|_2 \geq \sigma_{r+1}$.
4. Construct an E that achieves the above bound.

Exercise 5.8 Prove or disprove (through a counter example) the following singular values inequalities.

1. $\sigma_{min}(A+B) \leq \sigma_{min}(A) + \sigma_{min}(B)$ for any A and B.
2. $\sigma_{min}(A+E) \leq \sigma_{max}(E)$ whenever A does not have column rank, and E is any matrix.
3. If $\sigma_{max}(A) < 1$, then
$$\sigma_{max}(I-A)^{-1} \leq \frac{1}{1-\sigma_{max}(A)}$$
4. $\sigma_i(I+A) \leq \sigma_i(A) + 1$.