MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science

6.241: Dynamic Systems—Fall 2003

Homework 8 Solutions

Exercise 17.1 The system is not well posed if negative feedback is assumed. Consider the Figure 17.1 of the notes and replace the (+) with a (-) sign. After some algebraic manipulation one obtains

$$u(t) = -r(t+2) \quad t \geq 0$$

which corresponds to a non causal relationship.

Exercise 17.2 To check external stability of the interconnected (closed-loop) system we need to verify BIBO stability of the transfer functions from $\begin{pmatrix} w_1 & w_2 \end{pmatrix}^T$ to $\begin{pmatrix} y & u \end{pmatrix}^T$. The system can be represented by a transfer function matrix:

$$\begin{pmatrix} y \\ u \end{pmatrix} = \begin{pmatrix} (I+PK)^{-1}P & (I+PK)^{-1}PK \\ (I+KP)^{-1} & (I+KP)^{-1}K \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}.$$

Given

$$P = \begin{pmatrix} \frac{s}{s+1} & -\frac{\alpha}{s+1} \\ \frac{1}{s+1} & \frac{1}{s+1} \end{pmatrix}, \; K = \begin{pmatrix} \frac{s+1}{s(s+5)} & 0 \\ -\frac{s+1}{s(s+5)} & \frac{s+1}{s+5} \end{pmatrix}$$

we can see that

$$(I+PK)^{-1} = \begin{pmatrix} \frac{s(s+5)}{s^2+6s+\alpha} & \frac{\alpha s(s+5)}{(s+6)(s^2+6s+\alpha)} \\ 0 & \frac{s+5}{s+6} \end{pmatrix}$$

which is stable for $\alpha > 0$. Notice that P is stable as well, therefore $(I+PK)^{-1}P$ is stable for $\alpha > 0$. Other entries are given by:

$$(I+PK)^{-1}PK = \begin{pmatrix} \frac{s+\alpha}{s^2+6s+\alpha} & -\frac{\alpha s(s+5)}{(s+6)(s^2+6s+\alpha)} \\ 0 & \frac{1}{s+6} \end{pmatrix}, \; (I+KP)^{-1} = \begin{pmatrix} \frac{s+5}{s+6} & \frac{\alpha(s+5)}{(s+6)(s^2+6s+\alpha)} \\ 0 & \frac{s(s+5)}{s^2+6s+\alpha} \end{pmatrix}$$

$$(I+KP)^{-1}K = K(I+PK)^{-1} = \begin{pmatrix} \frac{s+1}{s^2+6s+\alpha} & \frac{\alpha(s+1)}{(s+6)(s^2+6s+\alpha)} \\ -\frac{s+1}{s^2+6s+\alpha} & \frac{s(s+1)}{s^2+6s+\alpha} \end{pmatrix}$$

We can see that for $\alpha > 0$ all of these transfer functions are stable, therefore the closed loop system is stable.

b) If $\alpha = 0$ the transfer function $(I+KP)^{-1}K$ will have poles at 0 (integrators), therefore the closed loop system is not stable.

Exercise 17.3 Given:

$$P(s) = \frac{1}{10s+1}, \quad K(s) = k,$$

with no measurement noise.

(i) In order for the feedback system to be stable, we would like to have the closed loop from r to y is stable. The closed loop transfer function from r to y is expressed as follows:

$$\begin{aligned} Y(s) &= (I+PK)^{-1}PKR(s) \\ &= \frac{PK}{1+PK} \quad \because \text{ SISO }. \\ &= \frac{\frac{K}{10s+1}}{1+\frac{K}{10s+1}}R(s) \\ \therefore Y(s) &= \frac{\frac{K}{10}}{s+\frac{1+K}{10}}R(s). \end{aligned}$$

Thus the feedback system is stable iff

$$\frac{1+K}{10} > 0 \rightarrow K > -1. \tag{1}$$

Since we are to find the least *positive* gain k, the first condition does not give us any information.

(ii) In order to evaluate $e(t)$, we need to have the closed loop transfer function from $R(s)$ to $E(s)$, which has the following form:

$$E(s) = \frac{1}{1+PK}R(s).$$

Yet, since the input of our interest here is a unit step, so that

$$R(s) = \frac{1}{s},$$

thus $E(s)$ has the following expression:

$$\begin{aligned} E(s) &= \frac{1}{PK}R(s) \\ &= \frac{1}{1+\frac{1}{10s+1}k}\frac{1}{s} \\ &= \frac{10s+1}{10s+1+k}\frac{1}{s}, \end{aligned}$$

with which we can use the final value theorem to evaluate $e(\infty)$ as follows:

$$\begin{aligned} e(\infty) &= \lim_{s\rightarrow 0} sE(s) \\ &= \lim_{s\rightarrow 0} s\frac{10s+1}{10s+1+k}\frac{1}{s} \\ &= \lim_{s\rightarrow 0} \frac{10s+1}{10s+1+k} \\ \therefore e(\infty) &= \frac{1}{1+k}, \end{aligned}$$

whose absolute value has to be less than 0.1. Thus

$$|\frac{1}{1+k}| \leq 0.1 \rightarrow K \geq 9, \tag{2}$$

since k has to be positive.

(iii) As indicated in the problem, first we would like to show that the $\mathcal{L}_2$ to $\mathcal{L}_\infty$ induced norm of a SISO system is given by $\mathcal{H}_2$ norm of the system. It can be proven as follows:

Here we have the following convolution relationship:

$$y(t) = h(t) * d(t),$$

then

$$\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} h(t-\tau)d(\tau)d\tau \\
\rightarrow |y(t)| &= |\int_{-\infty}^{\infty} h(t-\tau)d(\tau)d\tau| \\
&\leq \int_{-\infty}^{\infty} |h(t-\tau)||d(\tau)|d\tau \\
&\leq (\int_{-\infty}^{\infty} h(t-\tau)^2 d\tau)^{1/2}(\int_{-\infty}^{\infty} d(\tau)^2 d\tau)^{1/2}, \quad \because \text{ Cauchy-Schwartz inequality} \\
\therefore |y(t)| &\leq \|h\|_2\|d\|_2.
\end{aligned}$$

Thus by taking sup of both sides yields

$$\sup_t |y(t)| = \|y\|_\infty \leq \|h\|_2\|d\|_2.$$

Now we have to show that the equality can be achieved. Consider

$$d(t) = h(-t),$$

then clearly

$$\|d\|_2 = \|h\|_2.,$$

Thus, at $t = 0$,

$$\begin{aligned}
|y(0)| &= |\int_{\infty}^{\infty} h(0-\tau)h(-\tau)d\tau| \\
&= \|h\|_2^2 = \|h\|_2\|d\|_2.
\end{aligned}$$

Thus the equality can be achieved.

Hence the requirement can be interpreted as to have $\|h\|_2$, since

$$\|y\|_\infty \leq \|h\|_2\|d\|_2 \leq \|h\|_2 \leq 0.1$$

with $\|d\|_2 \leq 1$.
Here, the closed loop transfer function from $d(t)$ to $y(t)$ is given by

$$\begin{aligned} H(s) &= \frac{1}{1+PK} \\ &= \frac{1}{1+\frac{1}{10s+1}k} \\ &= 1 - \frac{k}{10s+1+k}. \end{aligned}$$

Since the transfer function is proper its $\mathcal{H}_2$ norm is infinite (this corresponds to the fact that one can always find a finite-energy input signal that will result in an output with infinite amplitude, take for example $t^{-1/4}$ for $0 \leq t \leq 1$, and 0 elsewhere) and, there exists no K such that the third requirement is met. Therefore, there exists no K to meet all the requirements.

Exercise 17.4 1) First, in order for the closed loop system to be stable, the transfer function from $(\begin{matrix} w_1 & w_2 \end{matrix})^T$ to $(\begin{matrix} y & u \end{matrix})^T$ has to be stable. The transfer function from w_1 to y is given by $(I - PK)^{-1}P$ and is called system response function. The transfer function from w_1 to u is given by $(I - KP)^{-1}$ and is called input sensitivity function. The transfer function from w_2 to y is $(I - PK)^{-1}PK$ and is called the complementary sensitivity function. The transfer function from w_2 to u is given by $(I - KP)^{-1}K$. Therefore, we have the following :

$$\begin{pmatrix} y \\ u \end{pmatrix} = \begin{pmatrix} (I+PK)^{-1}P & (I+PK)^{-1}PK \\ (I+KP)^{-1} & (I+KP)^{-1}K \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}.$$

So, if K is given as

$$K = Q(I - PQ)^{-1} = (I - QP)^{-1}Q,$$

then

$$\begin{aligned} (I+PK)^{-1}P &= (I + PQ(I-PQ)^{-1})^{-1}P \\ &= (((I-PQ)+PQ)(I-PQ)^{-1})^{-1}P \\ &= (I-PQ)P \\ (I+PK)^{-1}PK &= (I-PQ)PQ(I-PQ)^{-1} \\ &= P(I-QP)(I-QP)^{-1}Q \\ &= PQ \\ (I+KP)^{-1} &= (I+(I-QP)^{-1}QP)^{-1} \\ &= ((I-QP+QP)(I-QP)^{-1})^{-1} \\ &= I-QP \\ (I+KP)^{-1}K &= (I-QP)(I-QP)^{-1}Q \\ &= Q. \end{aligned}$$

Thus, the closed loop transfer function can be now written as follows:

$$\begin{pmatrix} y \\ u \end{pmatrix} = \begin{pmatrix} (I-PQ)P & PQ \\ (I-QP) & Q \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}.$$

In order for the closed loop system to be stable, then all the transfer functions in the large matrix above must be stable as well.

$$\begin{aligned}\|(I-PQ)P\| &\leq \|I-PQ\|\|P\| \leq (\|I\|+\|PQ\|)\|P\| \\ &\leq (\|I\|+\|P\|\|Q\|)\|P\| \leq \|P\|+\|P\|^2\|Q\| \\ \|PQ\| &\leq \|P\|\|Q\| \\ \|I-QP\| &\leq \|I\|+\|QP\| \leq \|I\|+\|Q\|\|P\|.\end{aligned}$$

Since P and Q are stable from the assumptions, we know that all the transfer functions are stable. Therefore the closed loop system is stable if $K = Q(I-PQ)^{-1} = (I-QP)^{-1}Q$.

2) From 1), we can express Q in terms of P and K in the following manner.

$$\begin{aligned}K &= Q(I-PQ)^{-1} \\ K(I-PQ) &= Q \\ K-KPQ &= Q \\ K &= (I+KP)Q \\ \rightarrow Q &= (I+KP)^{-1}K = K(I+PK)^{-1},\end{aligned}$$

by push through rule.

For *some* stable Q, the closed loop is stable for a stable P. by the stabilizing controller $K = Q(I-PQ)^{-1}$. Yet, *not all* stable Q can be used for this formulation because of the well-posedness of the closed loop. In the state space descriptions of P and Q, in order for the interconnected system, in this case $K(s)$ to be well-posed, we have to have the condition (17.4) in the lecture note, i.e., $(I - D_P Q(\infty))$ is invertible.

3) Suppose P is SISO, w_1 is a step, and $w_2 = 0$. Then, we have the following closed loop transfer function:

$$\begin{pmatrix} Y(s) \\ U(s) \end{pmatrix} = \begin{pmatrix} (I-PQ)P \\ I-QP \end{pmatrix} \frac{1}{s},$$

since the Laplace transform of the unit step is $\frac{1}{s}$ we have

$$U(s) = (1 - Q(s)P(s))\frac{1}{s}.$$

Then using the final value theorem, in order to have the steady state value of $u(\infty)$ to be zero, we need:

$$\begin{aligned}u(\infty) &= \lim_{s\rightarrow 0} s(1-Q(s)P(s))\frac{1}{s} = 0 \\ \rightarrow 1-Q(0)P(0) &= 0 \\ \rightarrow Q(0) &= 1/P(0).\end{aligned}$$

Therefore, $Q(0)$ must be nonzero and is equal to $1/P(0)$. Note that this condition implies that P cannot have a zero at $s = 0$ because then Q would have a pole at $s = 0$, which contradicts that Q is stable.

Exercise 18.2 Given :

$$G(s) = \begin{pmatrix} \frac{s-1}{s+1} & -5 \\ \frac{s+2}{(s+1)^2} & \frac{s-1}{s+1} \end{pmatrix}.$$

Then we would like to verify whether or not there exists a controller K such that $S = (I + PK)^{-1}$ satisfies $\|S\|_\infty < 1$.

1) No. Although G_{11} and G_{22} have nonminimum-phase zero at $s = 1$, it does not imply that the plant $G(s)$ has nonminimum-phase zeros. In order to find the zeros of $G(s)$, we have to find a variable coefficient $c(s)$ such that $G(s)$ loses its rank. It can be proceeded as follows:

$$\begin{pmatrix} \frac{s-1}{s+1} \\ \frac{s+2}{(s+1)^2} \end{pmatrix} = c(s) \begin{pmatrix} -5 \\ \frac{s-1}{s+1} \end{pmatrix},$$

from the first equation, we have $c(s) = -(1/5)\frac{s-1}{s+1}$. Substituting c(s) to the second equation yields

$$\begin{aligned} \frac{s+2}{(s+1)^2} &= -\frac{1}{5}\left(\frac{s-1}{s+1}\right)\left(\frac{s-1}{s+1}\right) \\ \rightarrow -5(s+2) &= (s-1)^2 \\ \rightarrow s^2 + 3s + 11 &= 0 \\ \therefore s &= \frac{-3 \pm i\sqrt{35}}{2}. \end{aligned}$$

Thus the zeros of the system $G(s)$ are minimum-phase zeros.

2) For this $G(s)$ we can indeed invert $G(s)$ and scale it in order to tune the sensitivity is uniformly less than one. $G(s)^{-1}$ is computed as follows:

$$\begin{aligned} G(s)^{-1} &= \frac{1}{\frac{s-1}{s+1}\frac{s-1}{s+1} + 5\frac{s+2}{(s+1)^2}} \begin{pmatrix} \frac{s-1}{s+1} & 5 \\ -\frac{s+2}{(s+1)^2} & \frac{s-1}{s+1} \end{pmatrix} \\ &= \frac{1}{s^2 + 3s + 11} \begin{pmatrix} (s+1)(s-1) & 5 \\ -(s+2) & (s+1)(s-1) \end{pmatrix}. \end{aligned}$$

Its poles are

$$s = \frac{-3 \pm \sqrt{35}}{2}.$$

Thus using the controller $K(s) = \alpha G(s)^{-1}$, $S(s)$, where $\alpha \in \mathbb{R}$, is expressed as follows:

$$S(s) = (I + GK)^{-1} = (I + \alpha I)^{-1} = \frac{1}{1+\alpha} I$$

Thus in order to have $\|S\|_\infty < 1$, we need the following condition on α :

$$\alpha > 0 \ , \ \alpha < -2.$$

Finally, check that this is indeed a stabilizing controller, and the right half plane zeros that we cancelled do not appear as right half plane poles in any of the transfer functions from all block inputs to all block outputs.