MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science

6.241: Dynamic Systems—Fall 2003

Homework 4 Solutions

Exercise 5.2 Using SVD, A can be decomposed as

$$A = U \begin{pmatrix} \sigma_1 & & & & & \\ & \ddots & & & & \\ & & \sigma_{r+1} & & & \\ & & & \ddots & & \\ & & & & \sigma_k & \\ & & & & & 0 \end{pmatrix} V',$$

where U and V are unitary matrices. Following the given procedure, let's select the first r+1 columns of V : $\{v_1, v_2, \cdots, v_{r+1}\}$. From the definition of SVD those v_i's are orthonormal, and hence independent. Note that $\{v_1, v_2, \cdots, v_{r+1}, \cdots v_n\}$ span $\mathbb{R}^n$, and if $rank(E) = r$, then exactly r of the vectors, $\{v_1, v_2, \cdots, v_{r+1}, \cdots v_n\}$, span $\mathcal{R}(E') = \mathcal{N}^{\perp}(E)$. The remaining vectors span $\mathcal{N}(E)$. So, given any $r+1$ linearly independent vectors in $\mathbb{R}^n$, at least one must be in the nullspace of E. That is, $\exists \alpha_i$ for $i = 1, \cdots, r+1$ not all zero such that

$$E(\alpha_1 v_1 + \alpha_2 v_2 + \cdots \alpha_{r+1} v_{r+1}) = 0.$$

This implies that there exists a nonzero vector z such that

$$z = \sum_{i=1}^{r+1} \alpha_i v_i = \begin{pmatrix} v_1 & \cdots & v_{r+1} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_{r+1} \end{pmatrix}$$

with $\|z\|_2 = 1$ such that $Ez = 0$. Thus,

$$(A - E)z = Az = U\Sigma \begin{pmatrix} - & {v_1}' & - \\ & \vdots & \\ - & {v_{r+1}}' & - \end{pmatrix} \left(\sum_{i=1}^{r+1} \alpha_i v_i \right) = U \begin{pmatrix} \sigma_1 \alpha_1 \\ \sigma_2 \alpha_2 \\ \vdots \\ \sigma_{r+1} \alpha_{r+1} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{1}$$

By taking 2-norm of both sides of the above equation,

$$
\begin{aligned}
\|(A-E)z\|_2 &= \|U \begin{pmatrix} \sigma_1\alpha_1 \\ \sigma_2\alpha_2 \\ \vdots \\ \sigma_{r+1}\alpha_{r+1} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \|_2 = \| \begin{pmatrix} \sigma_1\alpha_1 \\ \sigma_2\alpha_2 \\ \vdots \\ \sigma_{r+1}\alpha_{r+1} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \|_2 \quad \text{(since U is a unitary matrix)} \\
&= \left(\sum_{i=1}^{r+1} |\sigma_i \alpha_i|^2 \right)^{\frac{1}{2}} \geq \sigma_{r+1} \left(\sum_{i=1}^{r+1} |\alpha_i|^2 \right)^{\frac{1}{2}}. \qquad (2)
\end{aligned}
$$

But, from our construction of z,

$$
\|z\|_2^2 = 1 \rightarrow \|(\ v_1 \quad \cdots \quad v_{r+1} \ ) \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_{r+1} \end{pmatrix} \|_2^2 = 1 \rightarrow \| \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_{r+1} \end{pmatrix} \|_2^2 = \sum_{i=1}^{r+1} |\alpha_i|^2 = 1.
$$

Thus, equation(2) becomes

$$
\|(A-E)z\|_2 \geq \sigma_{r+1}.
$$

Finally, $\|(A-E)z\|_2 \leq \|A-E\|_2$ for all z such that $\|z\|_2 = 1$.

Now we have to achieve the lower bound. Choose

$$
E = U \begin{pmatrix} \sigma_1 & & & \\ & \ddots & & \\ & & \sigma_r & \\ & & & 0 \end{pmatrix} V'.
$$

E has rank r, and

$$
A - E = U \begin{pmatrix} 0 & & & & & & \\ & \ddots & & & & & \\ & & 0 & & & & \\ & & & \sigma_{r+1} & & & \\ & & & & \ddots & & \\ & & & & & \sigma_k & \\ & & & & & & 0 \end{pmatrix} V'.
$$

where $\|A-E\|_2 = \sigma_{r+1}$.

Exercise 5.8 (1) The statement is false, to construct a counterexample use $A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$.

(2) True, see exercise 5.2.

(3) True, first note that for any invertible matrix A we have $\sigma_{max}(A^{-1}) = \sigma_{min}(A)$, infact one can easily show that if $A \in \mathbf{C^{mxm}}$ then $\sigma_i(A) = \sigma_j(A^{-1})$ with $j = m - i + 1$ and $i \in \{1, ..., m\}$. Given

that remark all we need to show is that $1-\sigma_{max}(A) \leq \sigma_{min}(I-A)$. From the triangular inequality one has :

$$|\|Ix\|_2 - \|Ax\|_2| \leq \|(I-A)x\|_2$$

restricting our attention to unit vectors x, so $\|x\|_2 = 1$ and taking into account that $\sigma_{max}(A) < 1$ we obtain:

$$1 - \|Ax\|_2 \leq \|(I-A)x\|_2$$

but for any x on the unit sphere $1 - \sigma_{max}(A) \leq 1 - \|Ax\|_2$, thus

$$1 - \sigma_{max}(A) \leq \|(I-A)x\|_2$$

the above equation is still valid for any x on the unit sphere, ie $\|x\|_2 = 1$ so we can choose x such that $\|(I-A)x\|_2 = \sigma_{min}(I-A)$, thus

$$1 - \sigma_{max}(A) \leq \sigma_{min}(I-A)$$

completing the proof.

(4) Consider a matrix E with $rank(E) \leq i-1$, but otherwise arbitrary. From the triangular inequality we have

$$\|A+B-E\|_2 \leq \|A-E\|_2 + \|B\|_2$$

from exercise 5.2 we know that $\sigma_i(A) \leq \|A-E\|_2$ if $rank(E) = i-1$, thus

$$\sigma_i(A+B) \leq \|A-E\|_2 + \sigma_{max}(B)$$

now choose the particular E that achieves the bound, ie. $\|A-E\|_2 = \sigma_i(A)$ and we get

$$\sigma_i(A+B) \leq \sigma_i(A) + \sigma_{max}(B)$$

by letting $B = I$ we obtain the desired result.

Exercise 6.2 The model is nonlinear. To see this, consider two inputs, $u_1(t)$ and $u_2(t)$, with $u_1(t) > 1$ and $u_2(t) > 1$ such that $u_1(t) + u_2(t) > 1$. Then the output due to $u_1(t)$, $u_2(t)$, and $u_1(t) + u_2(t)$ is $y(t) = 1$.

The system is memoryless since the current output depends only on the current input.

Exercise 6.3 We know that when the input $u(t)$ is given as

$$u(t) = \begin{cases} 1 & \text{for } 1 \leq \text{t} \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

the corresponding output is

$$y(t) = \begin{cases} e^{t-1} - e^{t-2} & \text{for} \quad t \leq 1 \\ 2 - e^{1-t} - e^{t-2} & \text{for} \quad 1 \leq t \leq 2 \\ e^{2-t} - e^{1-t} & \text{for} \quad t \geq 2 \end{cases}$$

i) Assume that the map from $u(t)$ to $y(t)$, takes zero input for all time to the zero output for all time, *i.e.*, the input actually influences the output (there cannot be an output without an input). Recall that a system is causal if $P_T S P_T u = P_T S u$ for all T. Take $T = 1$ then,

$$P_1 S P_1 u = P_1 S(0) = 0$$

and

$$P_1 S u = P_1 y = e^{t-1} - e^{t-2}.$$

Since $P_1 S P_1 u \neq P_1 S u$, the system is not causal.

ii) The system is not memoryless either, since $y(\frac{1}{2}) = e^{\frac{-1}{2}} - e^{\frac{-3}{2}} \neq 0$ with $u(t) = 0$ for $t < 1$. In essence, the output $y(\frac{1}{2})$ depends on future values of the input.

Now suppose that the mapping is described by

$$y(t) = \int_{-\infty}^{\infty} e^{-|t-s|} u(s) ds.$$

iii) This system is linear. Let

$$y_1(t) = \int_{-\infty}^{\infty} e^{-|t-s|} u_1(s) ds$$

and

$$y_2(t) = \int_{-\infty}^{\infty} e^{-|t-s|} u_2(s) ds.$$

Then define the input as $\in \mathbb{C}$

$$u(t) = \alpha_1 u_1(t) + \alpha_2 u_2(t),$$

for any α_1, $\alpha_2 \in \mathbf{C}$, then the output is

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} e^{-|t-s|} (\alpha_1 u_1(s) + \alpha_2 u_2(s)) ds \\ &= \alpha_1 \int_{-\infty}^{\infty} e^{-|t-s|} u_1(s) ds + \alpha_2 \int_{-\infty}^{\infty} e^{-|t-s|} u_2(s) ds \\ &= \alpha_1 y_1(t) + \alpha_2 y_2(t), \end{aligned}$$

which shows that the system is linear.

iv) This system is time invariant. Consider an output $y_1(t)$ for an input $u_1(t) = u(t-T)$. Then

$$\begin{aligned} y_1(t) &= \int_{-\infty}^{\infty} e^{-|t-s|} u(s-T) ds \\ &= \int_{-\infty}^{\infty} e^{-|t-v-T|} u(v) dv \quad \text{(where } v = s-T\text{)} \\ &= \int_{-\infty}^{\infty} e^{-|(t-T)-v|} u(v) dv \\ &= y(t-T), \end{aligned}$$

which shows that $\forall T \in \mathbb{R}$, $u(t-T)$ results in output $y(t-T)$.

Exercise 7.1 a) Consider the input-output relationship (nonlinear difference equation)

$$y(k+n) = F(y(k+n-1), y(k+n-2), ..., y(k), u(k+n-1), u(k+n-2), ..., u(k), k).$$

This can equivalently be rewritten as the following difference equation:

$$y(k) = F(y(k-1), y(k-2), ..., y(k-n), u(k-1), u(k-2), ..., u(k-n), k-n).$$

Recall that in the LTI case, given a difference (or even a differential) equation, we can define states as delayed versions of the input and output. The same can be applied to the general non-linear time-varying case. Define

$$\begin{aligned} x_1(k-1) &= y(k-1) \\ &\vdots \\ x_n(k-1) &= y(k-n) \\ x_{n+1}(k-1) &= u(k-2) \\ &\vdots \\ x_{2n-1}(k-1) &= u(k-n). \end{aligned}$$

Then,

$$\begin{aligned} x_1(k) = y(k) &= F(\mathbf{x}(k-1), u(k-1), k-n) \\ x_2(k) &= x_1(k-1) \\ &\vdots \\ x_n(k) &= x_{n-1}(k-1) \\ x_{n+1}(k) &= u(k-1) \\ &\vdots \\ x_{2n-1}(k) &= x_{2n-2}(k-1) \end{aligned}$$

$$y(k) = x_1(k).$$

b) If the mapping F is LTI, then the difference equation has the form:

$$y(k) = a_1 y(k-1) + a_2 y(k-2) + ... + a_n y(k-n) + b_1 u(k-1) + b_2 u(k-2) + ...b_n u(k-n).$$

This can be rewritten as

$$y(k) = (a_1 y(k-1) + b_1 u(k-1)) + (a_2 y(k-2) + b_2 u(k-2)) + ... + (a_n y(k-n) + b_n u(k-n)).$$

As shown in lecture, we can realize the above difference equation with the block diagram shown in Figure 1. The states defined in the block diagram as the variables at the outputs of the shift operators, result in the following LTI state space description:

$$A = \begin{bmatrix} a_1 & 1 & 0 & ... & 0 \\ a_2 & 0 & 1 & ... & 0 \\ \vdots & 0 & 0 & 1 & ... \\ a_n & 0 & ... & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & ... & 0 \end{bmatrix}.$$

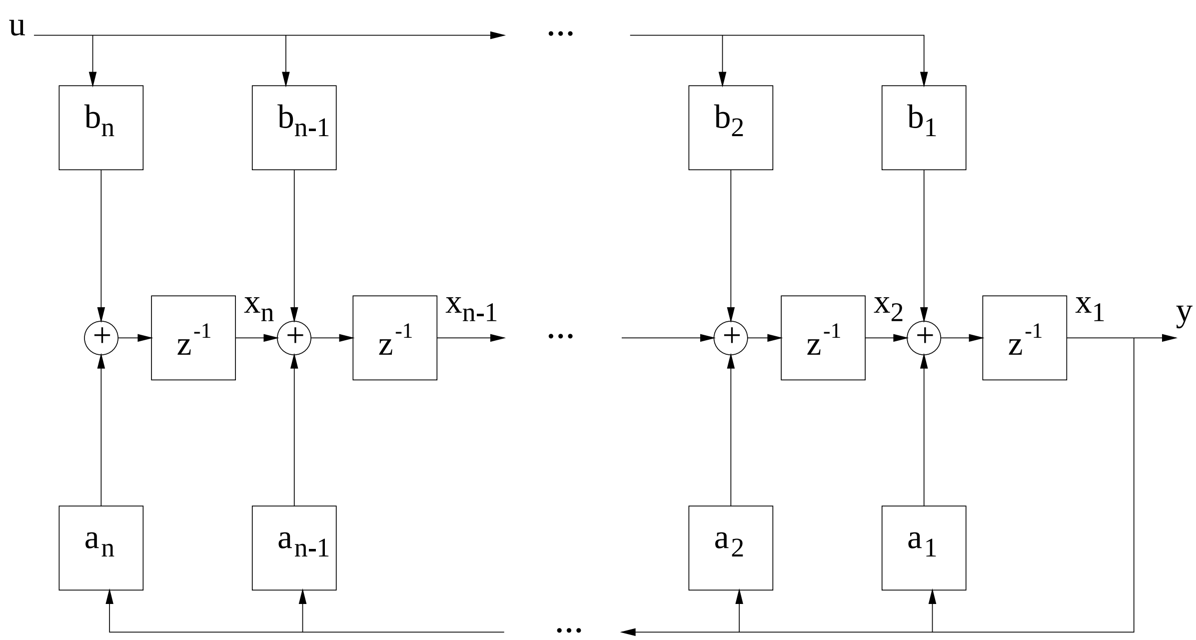

Figure 1: Block Diagram for the LTI difference equation in 7.1(b).

c) If F has the given form, note that the f_i are time invariant, the block diagram can be modified as shown in Figure 2. Now, write the state space description from the block diagram as follows:

$$\begin{aligned} x_n(k+1) &= f_n\left(u(k), y(k)\right) = f_n\left(x_1(k), u(k)\right) \\ x_{n-1}(k+1) &= x_n(k) + f_{n-1}\left(x_1(k), u(k)\right) \\ \dots & \quad \dots \\ x_1(k+1) &= x_2(k) + f_1\left(u(k), x_1(k)\right) \end{aligned}$$

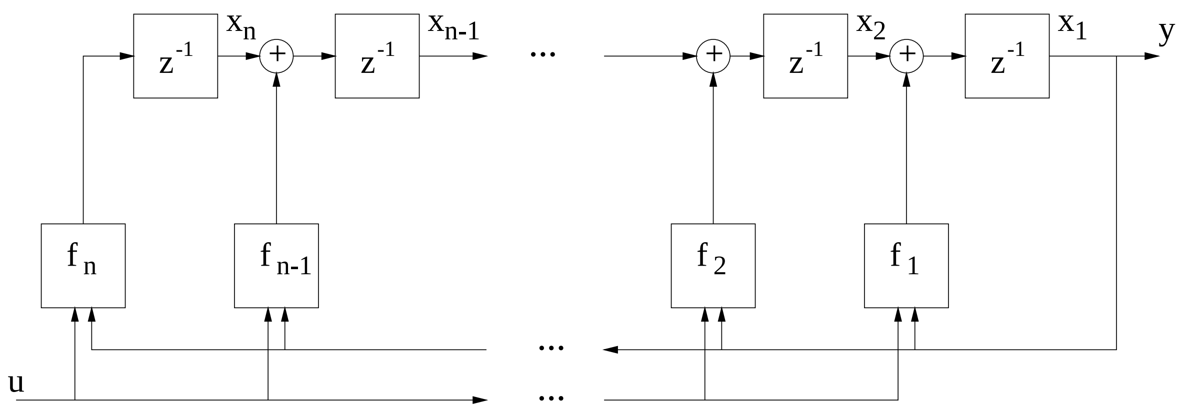

Figure 2: Block Diagram for Exercise 7.1(c).