MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science

6.241: Dynamic Systems—Fall 2003

HOMEWORK 11 SOLUTIONS

Exercise 22.1 Suppose x_1 is a reachable state, i.e., $x_1 = R_k U_{1_k}$ for some

$$U_{1_k} = \left[\begin{array}{cccc} u_1(0) & u_1(1) & \dots & u_1(k-1) \end{array}\right]'$$

where $u_1(i) \in [0,1]$ and $R_k = \left[\begin{array}{cccc} A^{k-1}B & A^{k-2}B & \dots & B \end{array}\right]$. Suppose that x_1 is in the half-plane H_1 where $H_1 = \{x \in \mathbb{R}^n | w'x \geq 0\}$ and $w'\lambda_1 = w'A$. We want to show that if x_2 is a reachable state, then $x_2 \in H_1$.

First, note that

$$w'x_1 = w' \sum_{i=0}^{k-1} A^{k-i-1} B u_1(i).$$

Furthermore, $w'A = w'\lambda_1$, so, $w'A^k = w'\lambda_1^k$, and

$$w'x_1 = \sum_{i=0}^{k-1} \lambda_1^{k-i-1} w' B u_1(i) = \left[\sum_{i=0}^{k-1} \lambda_1^{k-i-1} u_1(i) \right] w'B.$$

Now, since $x_1 \in H_1$ we have that $w'x_1 \geq 0$; and, since $\lambda_1 \geq 0$ and $u_1(i) \geq 0$ we must have that $w'B \geq 0$.

If x_2 is reachable, then $x_2 = R_l U_{2_l}$, where $u_2(i) \in [0,1]$. So,

$$w'x_2 = w' \sum_{i=0}^{l-1} A^{l-i-1} B u_2(i) = \left[\sum_{i=0}^{l-1} \lambda_1^{l-i-1} u_2(i) \right] w'B.$$

Since $w'B \geq 0$, $\lambda_1 \geq 0$, and $u_2(i) \geq 0$, we have that $w'x_2 \geq 0$, so $x_2 \in H_1$. Finally, we note that there was no loss in generality in assuming $x_1 \in H_1$, as opposed to $x_1 \in H_2 = \{x \in \mathbb{R}^n | w'x \leq 0\}$, because it can be shown that if $x_1 \in H_2$ then $x_2 \in H_2$, and the proof proceeds as above.

Exercise 22.3 a) The modal test is the most convenient in this case. The system is reachable if and only if $\text{rank}\,[\lambda I - A | B] = 5 \ \forall \lambda$ (need to check for λ equal to the eigenvalues of A). Observe that when $\lambda = 2$, $[\lambda I - A|B]$ is

$$\left[\begin{array}{cccccc} 0 & -1 & & & & b_1 \\ & 0 & & & & b_2 \\ & & 0 & & & b_3 \\ & & & -1 & -1 & b_4 \\ & & & & -1 & b_5 \end{array}\right],$$

which has rank 5 if and only if b_2 and b_3 are linearly independent. Similarly, $\lambda = 3$, $[\lambda I - A|B]$ is

$$\begin{bmatrix} 1 & 1 & & & & b_1 \\ & 1 & & & & b_2 \\ & & 1 & & & b_3 \\ & & & 0 & 1 & b_4 \\ & & & & 0 & b_5 \end{bmatrix},$$

which has rank 5 if and only if $b_5 \neq 0$.

(b) Suppose that $A \in \mathbb{R}^{n\times n}$ has k Jordan blocks of dimensions (number or rows) r_1, r_2, $\ldots$, r_k. Then we must have that $b_{r_1}, b_{r_1+r_2}, \ldots b_{r_1+r_2+r_3+\cdots+r_k} \neq 0$. Furthermore, if blocks r_i and r_j have the same eigenvalue, $b_{r_1+r_2+r_3+\cdots+r_i}$ and $b_{r_1+r_2+r_3+\cdots+r_j}$ must be linearly independent. These conditions imply that the input can excite the beginning of each Jordan chain, and hence has an impact on each of the states.

(c) If the $b_{i's}$ are scalars, then they are linearly dependent (multiples of each other), so if two of the Jordan blocks have the same eigenvalues the rank of $[\lambda I - A|B]$ is less than n.

Alternatively,

a) The system is reachable if none of the left eigenvectors of matrix A are orthogonal to B. Notice that to control the states corresponding to a Jordan block, it is sufficient to excite only the state corresponding the beginning of the Jordan chain, or the last element in the Jordan block (convince yourself of this considering a DT system for example). Thus it is not necessary that generalized eigenvectors are not orthogonal to the B matrix! Besides, notice that if two or more Jordan blocks have the same eigenvalue than any linear combination of eigenvectors corresponding to those Jordan blocks is a left eigenvector again. In case (a) we can identify left eigenvectors of matrix A:

$$\begin{aligned} w_2 &= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \end{bmatrix}' \\ w_3 &= \begin{bmatrix} 0 & 0 & 2 & 0 & 0 \end{bmatrix}' \\ w_5 &= \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix}' \end{aligned}$$

Any linear combination of w_2 and w_3 is also a left eigenvector. We can see that $w_k' B = b_k'$ - k^{th} row of matrix B. Therefore for reachability of matrix A we need to have at least one non-zero element in 5^{th} row and linear independence of 2^{rd} and 3^{rd} rows of matrix B.

b) Generalizing to an arbitrary matrix in Jordan form we can see that all rows of matrix B corresponding to a Jordan block with unique eigenvalue should have at least one non-zero element, and rows corresponding to Jordan blocks with repeated eigenvalues should be linearly independent.

c) If there are two or more Jordan blocks then we can find a linear combination of the eigenvectors which is orthogonal to the vector b, since two real numbers are obviously linearly dependent.

Exercise 22.4 The open loop system is reachable and has a closed-loop expression as follows:

$$x_{k+1} = Ax_k + B(w_k + f(x_k)),$$

where $f(\cdot)$ is an arbitrary but *known* function. Since the open loop system is reachable, there exists the control input u^* such that

$$u^* = \left(\begin{array}{ccc} u^*(0) & \cdots & u^*(n-1) \end{array} \right)^T .$$

that can drive the system to a target state in $\mathbb{R}^n$, $x_f = x^*(n)$. Thus let's define a trajectory $x^*(k)$ such that it starts from the origin and gets to $x^*(n)$ by the control input u^*. Then, since $u(k) = w(k) + f(x(k))$, let $w(k) = u^*(k) - f(x^*(K))$. Then this $w(k)$ can always take the system state from the origin to any specified target state in no more than n steps.

Exercise 23.1 a) We are given the single input LTI system:

$$\dot{x} = Ax + bu \ , \ A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \ , \ b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The solution is expressed by:

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)} bu(\tau)d\tau$$

Calculate exponent of matrix A by summing up the series and taking into account that $A^n = 0 \ , \ \forall n > 1$.

$$e^{At} = I + At = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

thus

$$e^{At} \ b = \begin{bmatrix} t \\ 1 \end{bmatrix}$$

b) the reachability matrix is:

$$\left[\begin{array}{cc} b & Ab \end{array} \right] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

The reachability matrix has rank 2, therefore the system is reachable. Now, we compute the reachability Grammian over an interval of length 1:

$$G = \int_0^1 e^{A(T-\tau)} bb' e^{A(T-\tau)'} d\tau = \begin{bmatrix} \frac{1}{3} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}$$

The system is reachable thus the Grammian is invertible, so given any final state x_f we can always find α such that $x_f = G\alpha$. In particular

$$\alpha = \frac{1}{\sqrt{2}} \begin{bmatrix} 18 \\ -10 \end{bmatrix}$$

c) According to 23.5 define $F^T(t) = e^{A(1-t)} \ b$. Then $u(t) = F(t)\alpha$ is a control input that produces a trajectory that satisfies the terminal constraint x_f. The control effort is given as:

$$\int_0^T u^2 d\tau = \alpha' \ G \ \alpha$$

Infact this input corresponds to the minimum energy input required to reach x_f in 1 second. This can be verified by solving the corresponding underconstrained least squares problem by means of the tools we learned at chapter 3.

d) First of all note that

$$\alpha' \ G \ \alpha = x_f' \ G^{-1} \ x_f$$

The Grammian as well as its inverse are symmetric matrices. If we want to maximize the energy, $\max\{x_f' G^{-1} x_f | \ \|x_f\| = 1\}$, we have to choose x_f alligned with the singular vector corresponding to $\sigma_{min}(G)$.

Exercise 23.4 Given :

$$\dot{x}(t) = Ax + (b + \delta)u,$$

where $\delta \in \mathbb{R}^n$, and (A, b) is reachable.

a) Using the Theorem 22.2, in order to make the system unreachable, we have $w^T B =$ for some left eigenvectors w^T of A. So, let λ_i is an eigenvalue of A and w_i be the corresponding left eigenvectors. Then, using the theorem, we want to find δ which makes this eigenmode unreachable $\leftrightarrow w_i^T(b + \delta) = 0$. So, now we have

$$w_i^T \delta = -w_i^T b.$$

Then with this constraint, we would like to minimize $\|\delta\|_2$. Thus this can be cast into an optimization problem as follows:

$$\begin{aligned} \text{Find} \quad & min\|\delta\|_2 \\ s.t. \quad & w_i^T \delta = -w_i^T b. \end{aligned}$$

This is exactly in the form of the least square problem. Since both δ and b are real, even when $w_i \in \mathbb{C}^n$, let $\tilde{w}_i = \begin{bmatrix} w_i^R & w_i^I \end{bmatrix}$, where w_i^R and w_i^I are real and imaginary parts of w_i respectively. Then the formulation still remains as a least square problem as follows:

$$\begin{aligned} \text{Find} \quad & \|\delta\|_2 \\ s.t. \quad & \tilde{w}_i^T \delta = \tilde{w}_i^T b. \end{aligned}$$

Then the solution to this problem is

$$\begin{aligned} \hat{\delta} &= -\tilde{w}_i(\tilde{w}_i^T \tilde{w}_i)^{-1} \tilde{w}_i^T b \\ \therefore \min \|\delta\|_2 &= \sqrt{\hat{\delta}^T \hat{\delta}} \end{aligned}$$

The last expression has to be minimized over all possible left eigenvectors of A. Note that the expression does not depend on the norm of the eigenvectors, thus we can minimize over eigenvectors with unity norm. If all Jordan blocks of matrix A have different eigenvalues, this is a minimization over a finite set. In the other case we can represent eigenvectors corresponding to Jordan blocks with the same eigenvalues as a linear combination of eigenvectors corresponding to particular Jordan blocks, and then minimize over the coefficients in the linear combination.

b) NO. The explanation is as follows. With the control suggested, the closed loop dynamics is now

$$\begin{aligned} \dot{x} &= Ax + (b+\delta)u \\ u &= f^T x + v \\ \rightarrow \dot{x} &= (A + (b+\delta)f^T)x + (b+\delta)v. \end{aligned}$$

Suppose that w_i was the minimizing eigenvector of unity norm in part a). Then it is also an eigenvector of matrix $A + (b+\delta)f^T$ since w_i is orthogonal to $b+\delta$. Therefore feedback does not improve reachability.