MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science

6.241: Dynamic Systems—Fall 2003

HOMEWORK 10 SOLUTIONS

Exercise 20.1 a) Given :

$$P_\alpha(s) = \begin{pmatrix} \frac{\alpha}{s-1} & \frac{1}{s-1} \\ \frac{2s-1}{s(s-1)} & \frac{1}{s-1} \end{pmatrix},$$

where $\alpha \in \mathbb{R}$ and in $[0.5, 1.5]$.

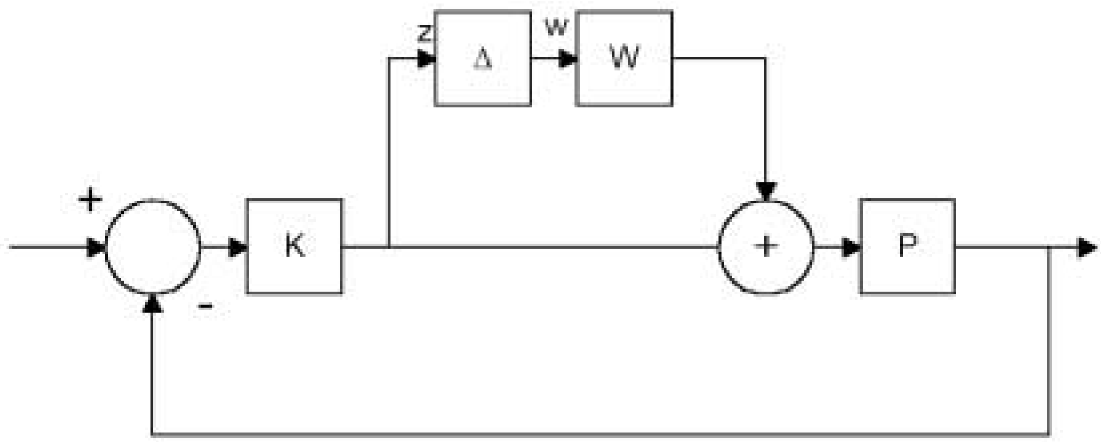

Figure 1: 20.1 Block Diagram.

In order to design a W which satisfies the condition given, first compute $P_{\tilde{\alpha}}^{-1}(s)(P_\alpha(s) - P_{\tilde{\alpha}}(s))$.

$$\begin{aligned}
P_\alpha(s) - P_{\tilde{\alpha}}(s) &= \frac{1}{s-1}\begin{pmatrix} \alpha - \tilde{\alpha} & 0 \\ 0 & 0 \end{pmatrix} \\
P_{\tilde{\alpha}}^{-1}(s) &= \frac{s(s-1)}{(\tilde{\alpha}-2)s+1}\begin{pmatrix} 1 & -1 \\ \frac{1-2s}{s} & \tilde{\alpha} \end{pmatrix}. \\
\rightarrow P_{\tilde{\alpha}}^{-1}(P_\alpha - P_{\tilde{\alpha}}) &= \frac{s(s-1)}{(\tilde{\alpha}-2)s+1}\begin{pmatrix} 1 & -1 \\ \frac{1-2s}{s} & \tilde{\alpha} \end{pmatrix}\begin{pmatrix} \alpha - \tilde{\alpha} & 0 \\ 0 & 0 \end{pmatrix}\frac{1}{s-1} \\
&= \frac{s}{(\tilde{\alpha}-2)s+1}\begin{pmatrix} \alpha - \tilde{\alpha} & 0 \\ \frac{(1-2s)}{s}(\alpha - \tilde{\alpha}) & 0 \end{pmatrix} \\
&= \frac{(\alpha - \tilde{\alpha})}{(\tilde{\alpha}-2)s+1}\begin{pmatrix} s & 0 \\ 1-2s & 0 \end{pmatrix}.
\end{aligned}$$

Thus, in order for the transfer function above to be stable, $\tilde{\alpha}$ needs to be more than 2, so the nominal plant $P_{\tilde{\alpha}}$ does not belong to the original set of plants. This is not a problem, since we are looking for a description that would include the original set, but not necessarily be one-to-one mapping.

Assume that $\tilde{\alpha} > 2$, let $A = P_{\tilde{\alpha}}^{-1}(P_\alpha - P_{\tilde{\alpha}})$, then $\sigma_{max}(A) =$ Eigenvalues of (A^*A). So, we need to find A^*A.

$$
\begin{aligned}
A^*(j\omega)A(j\omega) &= \left(\frac{(\alpha-\tilde{\alpha})}{(\tilde{\alpha}-2)j\omega+1}\begin{pmatrix} j\omega & 0 \\ 1-2j\omega & 0\end{pmatrix}\right)^* \left(\frac{(\alpha-\tilde{\alpha})}{(\tilde{\alpha}-2)j\omega+1}\begin{pmatrix} j\omega & 0 \\ 1-2j\omega & 0\end{pmatrix}\right) \\
&= \left(\frac{(\alpha-\tilde{\alpha})^2}{(-(\tilde{\alpha}-2)j\omega+1)((\tilde{\alpha}-2)j\omega+1)}\right)\begin{pmatrix} -j\omega & 1+2j\omega \\ 0 & 0\end{pmatrix}\begin{pmatrix} j\omega & 0 \\ 1-2j\omega & 0\end{pmatrix} \\
&\vdots \\
&= \frac{(\alpha-\tilde{\alpha})^2}{(\tilde{\alpha}-2)^2\omega^2+1}\begin{pmatrix} 1+5\omega^2 & 0 \\ 0 & 0\end{pmatrix}.
\end{aligned}
$$

Let us take $\tilde{\alpha} = 2.5$. To find a weighting function $W(s)$ which would serve as an upper bound for $\sigma_{max}(A(j\omega))$, let us use $\alpha = 0.5$. Then

$$W(s) = 4\frac{\sqrt{5}s+1}{0.5s+1},$$

To calculate Δ such that

$$P_1 = P_{2.5}\left[I + W\Delta\right]$$

we can directly invert the equation:

$$\Delta = W^{-1}P_{2.5}^{-1}[P_1 - P_{2.5}] = -\frac{3}{8(\sqrt{5}s+1)}\begin{bmatrix} s & 0 \\ 1-2s & 0\end{bmatrix}$$

We can see that Δ is stable, and calculating $\Delta^*\Delta$ we can show that $\|\Delta\|_\infty = 3/8 < 1$.
b) Set

$$W_1 = \begin{bmatrix} \frac{s}{\sqrt{5}s+1} \\ \frac{2s-1}{\sqrt{5}s+1} \end{bmatrix},\ W_2 = \begin{bmatrix} 4\frac{\sqrt{5}s+1}{0.5s+1} & 0 \end{bmatrix}$$

Then, the upper bound for $\sigma_{max}\left[P_{\tilde{\alpha}}^{-1}(P_\alpha - P_{\tilde{\alpha}})\right]$ can be given by the same weighting function as in a). The bound has a shape of a lead-lag compensator and its plot is given below:

c) For a): The uncertainty description given in a) is a multiplicative uncertainty so that it can be drawn as in Figure 5. So, the transfer function seen by Δ , w to z is written as follows:

$$
\begin{aligned}
z &= -KP_{\tilde{\alpha}}(Ww + z) \\
(I + KP_{\tilde{\alpha}})z &= -KP_{\tilde{\alpha}}Ww \\
z &= -(I + KP_{\tilde{\alpha}})^{-1}KP_{\tilde{\alpha}}Ww \\
&= -TWw \\
&= Mw.
\end{aligned}
$$

So, by the small gain theorem, we have

$$\|M\|_\infty = \|TW\|_\infty < 1.$$

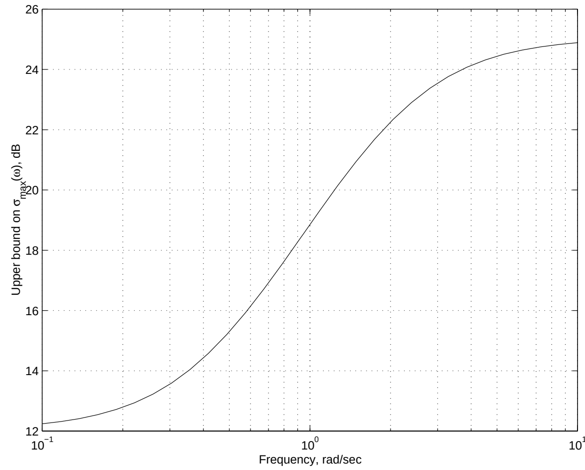

Figure 2: Upper bound on σ_{max}.

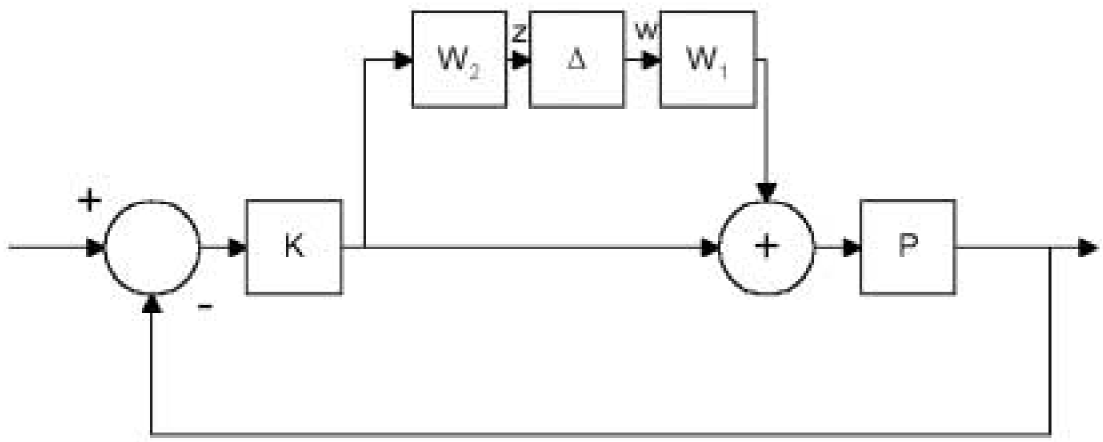

Figure 3: 20.1 c). Block Diagram for case b) .

For b): This case the block diagram can be drawn as follows:

Thus the transfer function from w to z is obtained as follows:

$$
\begin{aligned}
z &= W_2 v \\
v &= -KP_{\tilde{\alpha}}(W_1 w + v) \\
(I + KP_{\tilde{\alpha}})v &= -KP_{\tilde{\alpha}} W_1 w \\
v &= -(I + KP_{\tilde{\alpha}})^{-1} KP_{\tilde{\alpha}} W_1 w \\
\therefore z &= -W_2 (I + KP_{\tilde{\alpha}})^{-1} KP_{\tilde{\alpha}} W_1 w \\
&= -W_2 T W_1 w \\
&= Mw.
\end{aligned}
$$

Hence by the small gain theorem, we have

$$\|M\|_\infty = \|W_2 T W_1\|_\infty < 1.$$

Exercise 20.2 For stability, we need to check that $y_1 \in \mathcal{L}^q$ and $y_2 \in \mathcal{L}^m$ when $u_1 \in \mathcal{L}^m$ and $u_2 \in \mathcal{L}^q$. Here, $x \in \mathcal{L}^q$ means that x is a q-vector with bounded p-norm (we drop the subscript "p" from $\mathcal{L}_p$ for convenience). Suppose that $\|M\| = \gamma_1$, i.e., $\|y_1\| \leq \gamma_1 \|e_1\| \ \forall e_1 \in \mathcal{L}^m$ and $\|\Delta\| = \gamma_2$, i.e., $\|y_2\| \leq \gamma_2 \|e_2\| \ \forall e_2 \in \mathcal{L}^q$. Now, $e_1 = u_1 + y_2$ and $e_2 = u_2 + y_1$. So, $\|e_1\| \leq \|u_1\| + \|y_2\| \leq \|u_1\| + \gamma_2 \|e_2\|$ and $\|e_2\| \leq \|u_2\| + \|y_1\| \leq \|u_2\| + \gamma_1 \|e_1\|$. So, $\|e_1\| \leq \|u_1\| + \gamma_2 \|u_2\| + \gamma_1 \gamma_2 \|e_1\|$ and $\|e_2\| \leq \|u_2\| + \gamma_1 \|u_1\| + \gamma_1 \gamma_2 \|e_2\|$. Since $\gamma_1 \gamma_2 < 1$, we have $1 - \gamma_1 \gamma_2 > 0$, so, $\|e_1\| \leq \frac{1}{1-\gamma_1\gamma_2}(\|u_1\| + \gamma_2 \|u_2\|)$ and $\|e_2\| \leq \frac{1}{1-\gamma_1\gamma_2}(\|u_2\| + \gamma_1 \|u_1\|)$. Therefore, when $u_1 \in \mathcal{L}^m$ and $u_2 \in \mathcal{L}^q$, $\|u_1\|$ and $\|u_2\|$ are finite and so are $\|e_1\|$, $\|e_2\|$, $\|y_1\|$, and $\|y_2\|$.

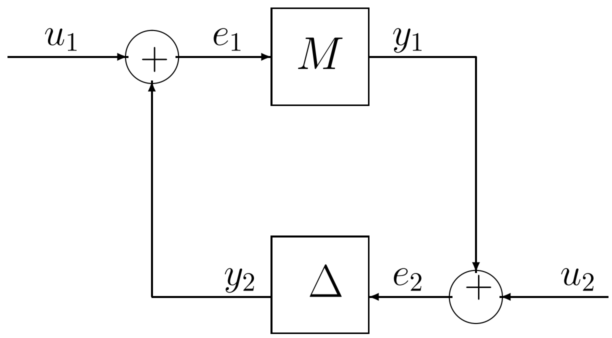

Figure 4: Problem 20.2 Block Diagram.

Exercise 21.1 We can use additive perturbation model with matrices W and Δ given in Figure 1.

$$W = \begin{bmatrix} W_{21} & 0 \\ 0 & W_{12} \end{bmatrix}, \ \Delta = \begin{bmatrix} 0 & \Delta_1 \\ \Delta_2 & 0 \end{bmatrix}, \ P_0 = \begin{bmatrix} P_{11} & 0 \\ 0 & P_{22} \end{bmatrix}, \ K = \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix}$$

Calculating transfer function from the output of Δ block w to its input z we get

$$M = \begin{bmatrix} 0 & \frac{W_{21}K_1\Delta_1}{1+K_{11}P_{11}} \\ \frac{W_{12}K_2\Delta_2}{1+K_{22}P_{22}} & 0 \end{bmatrix}$$

By assumption in the problem statement the decoupled system is stable, therefore the perturbed system will be stable if $I - M\Delta$ does not have zeros in the closed RHP for any Δ such that $\|\Delta_1\|_\infty \leq 1$ and $\|\Delta_2\|_\infty \leq 1$. By continuity argument this will be true if $I - M\Delta$ does not have zeros on $j\omega$ axis, or equivalently $|det\,(I - M(j\omega)\Delta(j\omega))| > 0$. Let us calculate the determinant in question:

$$\det\,(I - M\Delta) = \det \begin{bmatrix} 1 & \frac{W_{21}K_1\Delta_1}{1+K_{11}P_{11}} \\ \frac{W_{12}K_2\Delta_2}{1+K_{22}P_{22}} & 1 \end{bmatrix} = 1 - \frac{W_{12}W_{21}K_1K_2\Delta_1\Delta_2}{(1+K_{11}P_{11})\,(1+K_{22}P_{22})}$$

To have a stable perturbed system for arbitrary $\|\Delta_1\|_\infty \leq 1$ and $\|\Delta_2\|_\infty \leq 1$ it is necessary and sufficient to impose

$$\left| \frac{W_{12}W_{21}K_1K_2}{(1+K_{11}P_{11})\,(1+K_{22}P_{22})} \right| < 1$$

Since uncertainty blocks enter as a product, the answer will not change if only one block is perturbed.

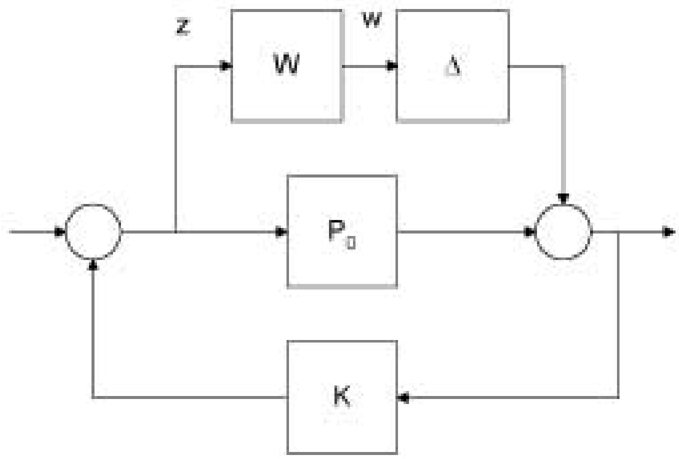

Figure 5: 21.1 Block Diagram.

Exercise 21.2

According to the lecture notes, we can state the problem equivalently as:

$$\mu(M) = \frac{1}{\inf_{\vec{\delta} \in \mathbb{R}^n}\{\max_i |\delta_i| : \sum_i \delta_i c_i = 1\}}$$

where $c_i = a_i \bar{b}_i$ and $M = \vec{a}\vec{b}'$.

This problem is substantially more difficult because the determination of $\inf_{\vec{\delta} \in \mathbb{R}^n}\{\max_i |\delta_i| : \sum_i \delta_i c_i = 1\}$ is equivalent to:

$$\inf_{\vec{\delta} \in \mathbb{R}^n}\{\max_i |\delta_i| : \Gamma\vec{\delta} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\}$$

where:

$$\Gamma = \begin{bmatrix} \mathcal{R}(\vec{c})' \\ \mathcal{I}(\vec{c})' \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

From this point on, we proceed for $n = 3$. The generelization to higher n is immediate.

Notice that the equality $\Gamma\vec{\delta} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ defines a line, say $L_0 \subset \mathbb{R}^3$, in $\mathbb{R}^3$. This makes our problem equivalent to finding the smallest β such that the cube $B_\beta = \{\vec{\delta} \in \mathbb{R}^n : \max_i |\delta_i| \leq \beta\}$ touches the line L_0. That can be done by looking at the following projections:

For every j such that $\mathcal{I}(c_j) \neq 0$, do the following: Look for the smallest β_j such that $\max_i |\delta_i| = \beta_j$ and

$$\sum_i \delta_i(\mathcal{R}(c_i) - \alpha_j \mathcal{I}(c_i)) = 1$$

where $\alpha_j = \frac{\mathcal{R}(c_j)}{\mathcal{I}(c_j)}$. For each such j, the example done in the lecture notes shows that the minimum β_j is $\frac{1}{\sum_i |\mathcal{R}(c_i) - \alpha_j \mathcal{I}(c_i)|}$ and the optimal $\vec{\delta}$ is $\delta_i = \beta_j sgn(\mathcal{R}(c_i) - \alpha_j \mathcal{I}(c_i))$. By doing that, we found the smallest β_j such that the projection of B_{β_j} in $\delta_j = 0$ touches the projection of L_0. Among all of the above candidate solutions, the only admissible is j^* such that $\delta_{j^*} = \frac{-\sum_{i \neq j^*} \mathcal{I}(c_i)\delta_i}{c_{j^*}}$ satisfies $|\delta_{j^*}| \leq \beta_{j^*}$.

The final solution is:

$$\mu(M) = \frac{1}{\beta_{j^*}} = \sum_i |\mathcal{R}(c_i) - \alpha_{j^*}\mathcal{I}(c_i)|$$

Exercise 21.3 The perturbed system can be represented by the diagram in Figure 2. The closed loop transfer function from reference input to output is

$$H(s) = \frac{N(s)}{D(s) + K(s)N(s)}$$

The system is stable if the denominator does not have zeros in the closed RHP.

$$D + KN = D_0 + KN_0 + D'_\delta\delta + KN'_\delta\delta = (D_0 + KN_0)\left(1 + \frac{D'_\delta\delta + KN'_\delta\delta}{D_0 + KN_0}\right)$$

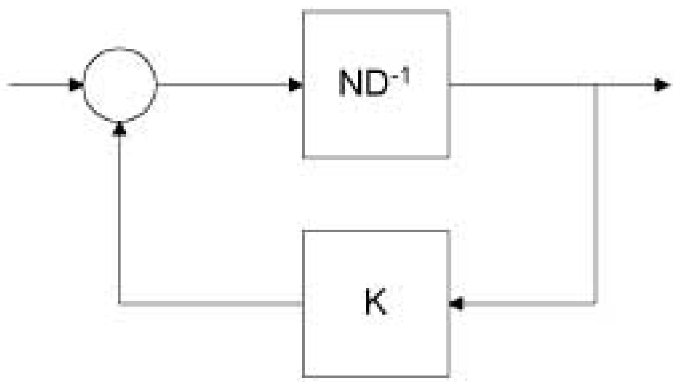

Figure 6: 21.3 Block Diagram.

From the statement of the problem we know that K is a stabilizing controller, which means that $D_0 + KN_0$ does not have zeros in the closed RHP. Therefore it is sufficient to show that the second parenthesis in the product above does not have zeros in the right half plane. By continuity argument we can see that the minimum norm δ at which stability is lost puts at least one root on $j\omega$ axis.

Now we can cast the problem as:

$$\inf_{\omega} \inf_{\delta \in \mathbb{R}^n} \{\max_i |\delta_i| : \frac{1}{D_0(j\omega) + KN_0(j\omega)} \left(D_\delta(j\omega) + KN_\delta(j\omega)\right)' \delta = -1\}$$

Now define the vector $c(\omega) = \frac{1}{D_0(j\omega)+KN_0(j\omega)} \left(D_\delta(j\omega) + KN_\delta(j\omega)\right)$, so that we can obtain the result, for every ω, using the previous problem:

$$\inf_{\delta \in \mathbb{R}^n} \{\max_i |\delta_i| : \frac{1}{D_0(j\omega) + KN_0(j\omega)} \left(D_\delta(j\omega) + KN_\delta(j\omega)\right)' \delta = -1\} = \frac{1}{\sum_i |\mathcal{R}(c_i(\omega)) - \alpha_{j^*}\mathcal{I}(c_i(\omega))|}$$