Exercises

Exercise 5.2 Let A and E be $m \times n$ matrices. Show that

$$\min_{\text{rank } E \leq r} \|A - E\|_2 = \sigma_{r+1}(A).$$

To prove this, notice that the rank constraint on E can be interpreted as follows: If $v_1, \ldots, v_{r+1}$ are linearly independent vectors, then there exists a nonzero vector z, expressed as a linear combination of such vectors, that belongs to the nullspace of E. Proceed as follows:

1. Select the v_i's from the SVD of A.
2. Select a candidate element z with $\|z\|_2 = 1$.
3. Show that $\|(A - E)z\|_2 \geq \sigma_{r+1}$. This implies that $\|A - E\|_2 \geq \sigma_{r+1}$.
4. Construct an E that achieves the above bound.

Exercise 5.8 Prove or disprove (through a counter example) the following singular values inequalities.

1. $\sigma_{min}(A + B) \leq \sigma_{min}(A) + \sigma_{min}(B)$ for any A and B.
2. $\sigma_{min}(A + E) \leq \sigma_{max}(E)$ whenever A does not have column rank, and E is any matrix.
3. If $\sigma_{max}(A) < 1$, then
$$\sigma_{max}(I - A)^{-1} \leq \frac{1}{1 - \sigma_{max}(A)}$$
4. $\sigma_i(I + A) \leq \sigma_i(A) + 1$.

Exercise 6.2 Suppose the input-output relation of a system is given by

$$y(t) = \begin{cases} u(t) & \text{if } |u(t)| \leq 1 \\ \frac{u(t)}{|u(t)|} & \text{if } |u(t)| > 1 \end{cases}.$$

This input-output relation represents a *saturation* element. Is this map nonlinear? Is it memoryless?

Exercise 6.3 Consider a system modeled as a map from $u(t)$ to $y(t)$, and assume you know that when

$$u(t) = \begin{cases} 1 & \text{for } 1 \le t \le 2 \\ 0 & \text{otherwise} \end{cases},$$

the corresponding output is

$$y(t) = \begin{cases} e^{t-1} - e^{t-2} & \text{for } t \le 1 \\ 2 - e^{1-t} - e^{t-2} & \text{for } 1 \le t \le 2 \\ e^{2-t} - e^{1-t} & \text{for } t \ge 2 \end{cases}.$$

In addition, the system takes the zero input to the zero output. Is the system causal? Is it memoryless?

A particular mapping that is consistent with the above experiment is described by

$$y(t) = \int_{-\infty}^{\infty} e^{-|t-s|} u(s) ds. \qquad (6.24)$$

Is the model linear? Is it time-invariant?

Exercise 7.1 Consider the nonlinear difference equation

$$y(k+n) = F[y(k+n-1), \ldots, y(k), u(k+n-1), \ldots, u(k), k]$$

where n is a fixed integer, and k is the time index.

(a) Find a state-space representation of order $2n-1$ for this difference equation.

(b) Find an nth-order state-space representation in LTI case (what is the form of F in this case?), using z-transforms for guidance (natural state variables are the coefficients of the initial-condition terms in the z-transformed version of the difference equation — try a third-order difference equation — remind of forward shift theorem from z-transforms). This part will guide the solution of (c).

(c) Find an nth-order state-space representation for the nonlinear system in (a) for the case where $F[\,.\,]$ has the special form

$$F[\,.\,] = \sum_{i=1}^{n} f_i[y(k+n-i), u(k+n-i)]$$

(Hint: Note that the difference equation in part (b) has this form; use your definition of state variables in (b) to guide your choice here.)