6.241 Recitation 8

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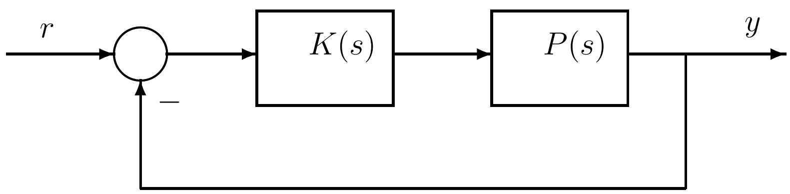
Nyquist Stability Criterion

In this recitation, the Nyquist stability criterion for SISO LTI systems is reviewed.

1 Stability of Feedback Systems

The objective is to derive conditions on the stability of the closed loop system by studying the open loop system.



For simplicity, suppose that both plant and controller are strictly proper, the conclusions will not change in the more general case but the algebra is cumbersome.

$$\begin{aligned} \dot{x}_p &= A_p x_p + B_b u_p \\ y_p &= C_p x_p \end{aligned} \qquad \begin{aligned} \dot{x}_k &= A_k x_k + B_b u_k \\ y_k &= C_k x_k \end{aligned}$$

The open loop transfer function is $L(s) = P(s)K(s)$, in state space representation:

$$\begin{bmatrix} \dot{x}_p \\ \dot{x}_k \end{bmatrix} = \underbrace{\begin{bmatrix} A_p & B_p C_k \\ 0 & A_k \end{bmatrix}}_{A_{OL}} \begin{bmatrix} x_p \\ x_k \end{bmatrix} + \underbrace{\begin{bmatrix} 0_p \\ B_k \end{bmatrix}}_{B} u$$

$$y = \underbrace{\begin{bmatrix} C_p & 0 \end{bmatrix}}_{C} \begin{bmatrix} x_p \\ x_k \end{bmatrix}$$

The state space description of the closed loop system is:

$$\begin{bmatrix} \dot{x}_p \\ \dot{x}_k \end{bmatrix} = \underbrace{\begin{bmatrix} A_p & B_p C_k \\ -B_k C_p & A_k \end{bmatrix}}_{A} \begin{bmatrix} x_p \\ x_k \end{bmatrix} + \begin{bmatrix} 0_p \\ B_k \end{bmatrix} u$$

$$y = \begin{bmatrix} C_p & 0 \end{bmatrix} \begin{bmatrix} x_p \\ x_k \end{bmatrix}$$

Let $\Phi_{OL}(s)$ and $\Phi_{CL}(s)$ be the open loop and the closed loop characteristic polynomials respectively.

$$\Phi_{OL} = det(sI - A_{OL})$$

$$\begin{aligned} \Phi_{CL}(s) &= det(sI - A) = det(sI - A_{OL} + BC) \\ &= det\left[(sI - A_{OL}) \cdot (I + (sI - A_{OL})^{-1}BC)\right] \\ &= det(sI - A_{OL}) \cdot det(I + \underbrace{C(sI - A_{OL})^{-1}B}_{L(s)}) \\ &= \Phi_{OL}(s) \cdot det(I + L(s)) \end{aligned}$$

The closed loop system poles are the zeros of Φ_{CL}.

2 The Principle of the Argument

Consider a complex-valued function on **C**

$$f(s) \; : \; \mathbf{C} \rightarrow \mathbf{C}$$

and a closed contour $\mathcal{C}$ in **C**. Suppose:

1) $f(s)$ has no poles or zeros on $\mathcal{C}$

2) $f(s)$ has at most a finite number of poles within $\mathcal{C}$, i.e. $f(s)$ is analytic within $\mathcal{C}$ except for at most a finite number of points.

Then $f(s)$ maps the closed contour $\mathcal{C}$ into a closed contour $\mathcal{C}'$ that includes the origin of the $f(s) - plane$. Moreover, if we call

N the number of clockwise encirclements of the origin of the $f(s) - plane$ by $\mathcal{C}'$, as s traverses $\mathcal{C}$ clockwise once,

Z the number of zeros of $f(s)$ in $\mathcal{C}$

P the number of poles of $f(s)$ in $\mathcal{C}$

then

$$N = Z - P.$$

Note positive N corresponds to clockwise encirclements.

The next fact presents a useful result when $f(s)$ in $\mathcal{C}$ is represented by the product of two other functions $f_1(s)$ and $f_2(s)$ satisfying assumptions 1) and 2) above. The principle of the argument will be use several times in what follow. To simplify its statements we introduce the following notation.

$$N(0, f(s), \mathcal{C}).$$

This stands for "the number of clockwise encirclements of the point 0 by $f(s)$ when s traverses the contour $\mathcal{C}$ clockwise once. By the Principle of the Argument we have that

$$N(0, f(s), \mathcal{C}) = Z - P$$

Fact 2.1 *Suppose $f(s) = f_1(s)f_2(s)$ with $f_1(s)$ and $f_2(s)$ satisfying the assumptions* 1) *and* 2) *above for a contour $\mathcal{C}$. Suppose, moreover, that*

$f_1(s)$ has Z_1 zeros and P_1 poles in $\mathcal{C}$.

$f_2(s)$ has Z_2 zeros and P_2 poles in $\mathcal{C}$.

Then, $\quad N(0, f(s), \mathcal{C}) = N(0, f_1(s), \mathcal{C}) + N(0, f_2(s), \mathcal{C}) = Z_1 - P_1 + Z_2 - P_2$
Note that $f_1(s)$ and $f_2(s)$ are allowed to have common poles and zeros.

Fact 2.2

$$N(0, f(s), \mathcal{C}) = N(-a, a + f(s), \mathcal{C})$$

for any $a \in \mathbf{C}$

We want to apply the Principle of the Argument and Fact 2.1 to derive a condition that ensures that $\Phi_{CL}(s)$ has no zeros in the RHP.
In order to do this, we need to define an appropriate contour which contains the whole RHP.

3 The D Contour

Poles of $f(s)$ on the $j - \omega$ axis must be either avoided or included by the contour. When avoided, they do not count as poles in RHP.

The $D - contour$ is defined for $R \to \infty$ and $\epsilon \to 0$.
Recall that

$$\Phi_{CL}(s) = \Phi_{OL}(s) \cdot det(I + L(s))$$

We now apply the Principle of the Argument to $\Phi_{CL}(s)$.

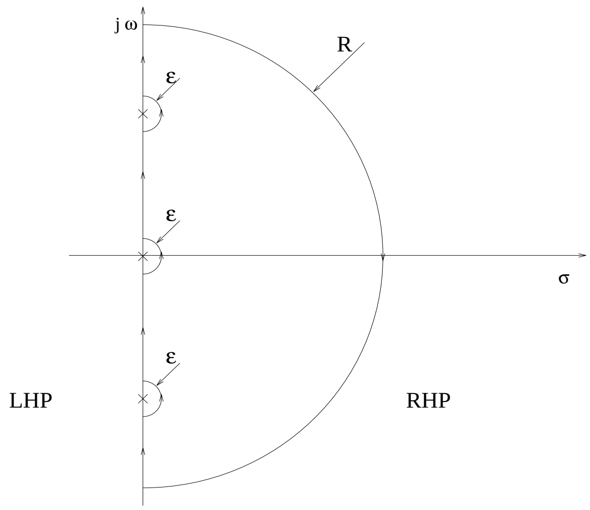


Figure 1: Sketch of D Contour

4 Nyquist Criterion

The Closed loop system is stable if and only if

$$N(0, det(I + L(s)), D) = -N(0, \Phi_{OL}(s), D) = -P_u$$

where P_u is the number of the unstable poles of the open loop map.
Proof.

$$Z = N(0, \Phi_{CL}(s), D)$$

P in this case is zero since $\Phi_{CL}(s)$ is a polynomial, (it has no poles at all). From Fact 2.1 then it follows that

$$Z = N(0, \Phi_{OL}(s), D) + N(0, det(I + L(s)), D)$$

But the zeros of $\Phi_{OL}(s)$ are the unstable open loop poles therefore:

$$N(0, \Phi_{OL}(s), D) = P_u$$

For the stability of the closed loop system we want that $\Phi_{CL}(s)$ has no zeros in RHP, hence Z=0 and:

$$N(0, det(I + L(s)), D) = -P_u$$

In words: the closed loop system is stable if an only if the number of counterclockwise encirclements of the origin by $det(I + L(s))$ when s traverses the $D - contour$ clockwise once, must be equal to the number of open loop unstable poles.

Note: In the above form the Nyquist Criterion is valid even if there are unstable pole/zero cancellation between the controller and the plant as far as the cancelling unstable poles are counted in P_u.

If there are not unstable pole/zero cancellations between the controller and the plant, then P_u is equal to the number of unstable poles of $det(I + L(s))$.

4.1 The SISO case

When $L(s)$ is a SISO transfer function, then

$$det(I + L(s)) = 1 + L(s)$$

Using Fact 2.2 the Nyquist Criterion can be restate in the its classical form, namely:

$$N(-1, L(s), D) = -P_u$$

Therefore, the closed loop is stable is and only if the number of counterclockwise encirclements of the point -1 $L(s)$ is equal to the number of open loop unstable poles.
Note: in general there is no usefulness in stating the criterion for MIMO $L(s)$ as

$$N(-1, -1 + det(I + L(s)), D) = -P_u$$

but sometimes it may simplify the computations.

4.2 Characteristics of the Nyquist Plot

$$L(s) = P(s)K(s)$$

A general form for a SISO $L(s)$ is

$$L(s) = \frac{k}{s^m} \frac{\prod_i (s + z_i)}{\prod_j (s + p_j)} = \frac{L'(s)}{s^m}$$

Behavior near $\omega = 0$

$$|L(\omega)| \simeq \frac{|L'(0)|}{w^m}; \qquad \angle L(\omega) \simeq -\frac{\pi}{2} m - \angle L(0)$$

Behavior for $\omega \to \infty$ for strictly proper $L(s)$. Let

$$q = \# \text{ poles of } L(s) - \# \text{ zeros of } L(s)$$

For ω very large

$$|L(\omega)| \simeq \frac{k}{\omega^q}; \qquad \angle L(\omega) \simeq -\frac{\pi}{2} q - \angle k$$

Symmetry with respect to the real axis: $L(s^*) = L^*(s)$

4.3 Examples

1)

$$L(s) = \frac{s+1}{s+2}$$

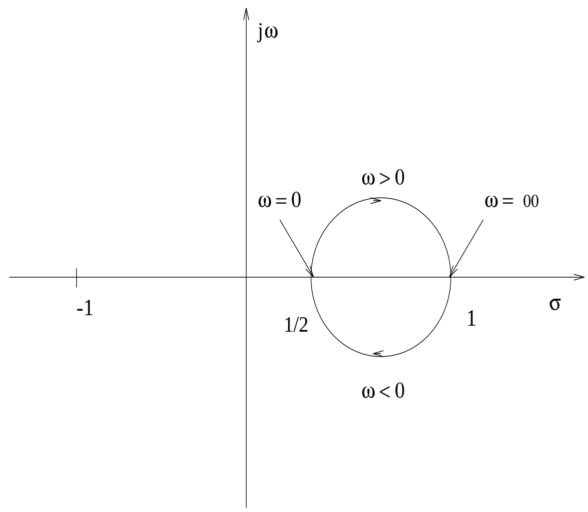


Figure 2: Sketch of L(s) Along D Contour

$Z = N + P_u$, $P_u = 0$, ($L(s)$ stable), $N = 0$, therefore $Z = 0$, closed loop system stable.

2)

$$L(s) = \frac{k}{s+1}$$

We want to study the stability of the closed loop system as k varies from $-\infty$ to ∞.

- For $k > 0$ $N = 0$, $P_u = 0$ the closed loop system is stable.
- For $k > -1$ $N = 0$, $P_u = 0$ the closed loop system is stable.
- For $k \leq -1$ $N = 1$, but $P_u = 0$, therefore the closed loop system will have one unstable pole.

4.4 A Special Property of the SISO Nyquist Plot

Motivated by the last example, suppose we want to study the stability of the closed loop when the open loop transfer function is $L'(s) = \alpha L(s)$ where α is a constant positive gain. It is easy to see that

$$N(-1, L'(s), D) = N(-\frac{1}{\alpha}, L(s), D)$$

Therefore one can plot the Nyquist diagram for $\alpha = 1$ and study the encirclements of the point $1/\alpha$. Unfortunately this property does not hold for the MIMO case and it limits the

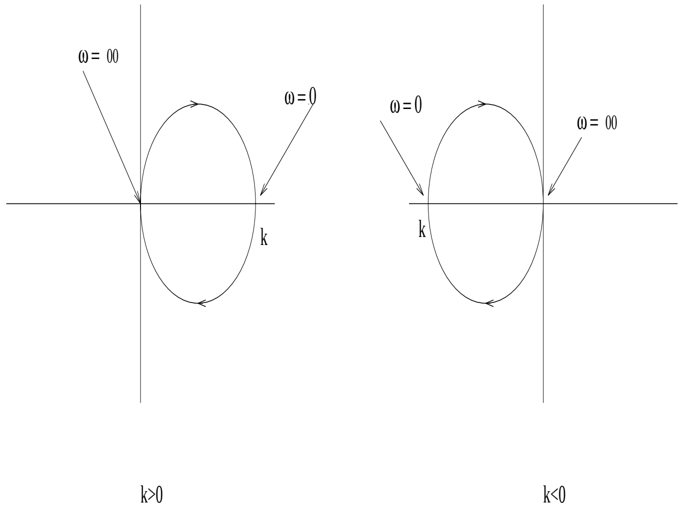


Figure 3: Sketches for Example 2

usefulness of the Nyquist diagram. The fact that for a MIMO system, the Nyquist plot passes close to the origin, in general, it is not an indication of robustness against perturbations of the open loop gain.