

**6.241: Dynamic Systems—Fall 2003**

RECITATION 7

## Input-Output Stability

In this recitation, we review the  $\infty$ -norm, 1-norm, and 2-norm on the vector spaces  $\mathcal{L}_p^n(\mathbb{R})$ , where  $p = \infty, 1$ , or  $2$  respectively (or  $\ell_p^n$  for discrete time signals) discussed in lecture. The Fourier Transform is defined and a proof for Parseval's relation is presented.

### $\mathcal{L}_p$ Spaces

A dynamic system could be thought of as an operator that acts on a set of signals, just as a  $m \times n$  matrix is an operator that acts on the vector space  $\mathbb{R}^n$ . We are interested in considering the effect of our dynamic system on particular sets of signals, specifically, vector spaces whose elements are certain signals and on which we can define a notion of the size of a signal (i.e. a norm). Once we have defined our vector space and norm, we can define a notion of size or “maximum amplification” of the dynamic system, just as we had done for matrices (recall that the induced norm was a measure of maximum amplification of a matrix). The size or induced norm of the system will depend on the norm we pick, so we quickly review signal norms below.

Consider the following sets of signals:

	Continuous Time $\mathcal{L}_p^n(\mathbb{R})$	Discrete Time $\ell_p^n(\mathbb{Z})$
$p = \infty$	$\{f : \mathbb{R} \rightarrow \mathbb{R}^n \mid \sup_t \ f(t)\ _\infty < \infty\}$	$\{f : \mathbb{Z} \rightarrow \mathbb{R}^n \mid \sup_k \ f(k)\ _\infty < \infty\}$
$p = 1$	$\{f : \mathbb{R} \rightarrow \mathbb{R}^n \mid \int_{\mathbb{R}} \ f(t)\ _1 dt < \infty\}$	$\{f : \mathbb{Z} \rightarrow \mathbb{R}^n \mid \sum_{k \in \mathbb{Z}} \ f(k)\ _1 < \infty\}$
$p = 2$	$\{f : \mathbb{R} \rightarrow \mathbb{R}^n \mid \int_{\mathbb{R}} \ f(t)\ _2^2 dt < \infty\}$	$\{f : \mathbb{Z} \rightarrow \mathbb{R}^n \mid \sum_{k \in \mathbb{Z}} \ f(k)\ _2^2 < \infty\}$

Check that each of these is a vector space. That is, if  $f, g \in \mathcal{L}_p$ , then  $f + g \in \mathcal{L}_p$  and  $\alpha f \in \mathcal{L}_p$ , where  $\alpha$  is any scalar (note that this latter condition includes the condition that the zero vector must belong to the vector space; just let  $\alpha$  equal zero). Furthermore, you can verify that the following are valid norms for each of the above spaces:

		Continuous Time $\mathcal{L}_p^n(\mathbb{R})$	Discrete Time $\ell_p^n(\mathbb{Z})$
Peak Magnitude	$p = \infty$	$\sup_t \ f(t)\ _\infty$	$\sup_k \ f(k)\ _\infty$
Action	$p = 1$	$\int_{\mathbb{R}} \ f(t)\ _1 dt$	$\sum_{k \in \mathbb{Z}} \ f(k)\ _1$
Energy	$p = 2$	$[\int_{\mathbb{R}} \ f(t)\ _2^2 dt]^{\frac{1}{2}}$	$[\sum_{k \in \mathbb{Z}} \ f(k)\ _2^2]^{\frac{1}{2}}$

Finally, we note that although the power of a signal,  $P_f = \lim_{T \rightarrow \infty} (\frac{1}{2T} \int_{-T}^T f(\tau)^T f(\tau) d\tau)$  (CT) and  $P_f = \lim_{N \rightarrow \infty} (\frac{1}{2N} \sum_{k=-(N-1)}^{N-1} f(k)^T f(k))$  (DT), is not a norm, because a nonzero signal can have zero power, power is a useful signal measure.

**Example 1:** Consider the following continuous time signal

$$w(t) = \begin{cases} 1 & \text{if } t \in [2^{2k}, 2^{2k+1}], \text{ for } k = 1, 2, \dots \\ 0 & \text{otherwise} \end{cases}$$

By inspection, we have that  $\|w\|_\infty = 1$  and  $\|w\|_1 = \|w\|_2 = \infty$ . Furthermore,  $P_w$  does not exist. This can be concluded as follows. First note that power is the limit of  $f(T)$  as  $T \rightarrow \infty$  where  $f(T) = (\frac{1}{2T} \int_{-T}^T w(\tau)^T w(\tau) d\tau)$ . On one hand, for  $f(2^{2N+1})$ ,

$$f(2^{2N+1}) = \frac{1}{2 \cdot 2^{2N+1}} \sum_{k=1}^N 2^{2k+1} - 2^{2k} = \frac{1}{4} \sum_{k=1}^N 4^{k-N} \rightarrow \frac{1}{4} \frac{1}{1 - \frac{1}{4}} = \frac{1}{3} \text{ as } N \rightarrow \infty$$

And on the otherhand,

$$f(2^{2(N+1)}) = \frac{1}{2} f(2^{2N+1}) \rightarrow \frac{1}{6} \text{ as } N \rightarrow \infty$$

That is, the power of the signal does not exist, because the integral over a length  $2T$  oscillates between 2 fixed values as  $T \rightarrow \infty$ .

**Example 2:** Prove that for discrete time signals,  $w(k)$  (for simplicity consider scalar-valued signals, *i.e.*,  $w(k) : \mathbb{Z} \rightarrow \mathbb{R}$ .) that are bounded in the  $p$  norm are bounded in the  $p+q$  norm for  $q > 0$ .

We are given that  $w \in \ell_p$  and we want to show that  $w \in \ell_{p+q}$ . That is, we know that  $\sum_k |w(k)|^p < \infty$  and we want to show that  $\sum_k |w(k)|^{p+q} < \infty$ . Let's split the domain of  $w(k)$  into two sets,  $K_0 = \{k \in \mathbb{Z} | |w(k)| < 1\}$  and  $K_1 = \{k \in \mathbb{Z} | |w(k)| \geq 1\}$ . So,  $\sum_k |w(k)|^p = \sum_{k \in K_0} |w(k)|^p + \sum_{k \in K_1} |w(k)|^p < \infty$ , from which we can conclude that  $\sum_{k \in K_0} |w(k)|^p < \infty$  and  $\sum_{k \in K_1} |w(k)|^p < \infty$ . The latter inequality implies that signals in  $\ell_p$  must have a finite number of elements  $w(k)$  with magnitude greater than or equal to 1. This in turn implies that  $\sum_{k \in K_1} |w(k)|^{p+q} < \infty$ . Now recall that  $\|w\|_{p+q}^{p+q} = \sum_{k \in K_0} |w(k)|^{p+q} + \sum_{k \in K_1} |w(k)|^{p+q}$ , so it remains to show that  $\sum_{k \in K_0} |w(k)|^{p+q} < \infty$ . Recall that for  $k \in K_0$ ,  $|w(k)| < 1$ . Thus,  $|w(k)|^{p+q} < |w(k)|^p$ . This implies that  $\sum_{k \in K_0} |w(k)|^{p+q} < \sum_{k \in K_0} |w(k)|^p < \infty$ , which is what we want.

**Example 3:** In general, for continuous time signals, the  $\mathcal{L}_p$  spaces are different spaces. For example, the signal  $f(t) = t^{-3/4}$  on  $t \in (0, 1]$  and zero elsewhere is in  $\mathcal{L}_1(\mathbb{R})$  but not in  $\mathcal{L}_2(\mathbb{R})$ . Conversely, the signal  $f(t) = t^{-3/4}$  on  $t \in [1, \infty)$  and zero elsewhere is in  $\mathcal{L}_2(\mathbb{R})$  but not in  $\mathcal{L}_1(\mathbb{R})$ .

## System Norms

Now that we have defined some measures of signals, we can define the notion of an input-output stable system  $H$ . A system  $H$  (not necessarily finite dimensional) is *p-stable* if there exists a constant  $C_p \in \mathbb{R}$  such that

$$\|y\|_p \leq C_p \|u\|_p.$$

The smallest such constant  $C_p$  is called the *induced norm* of the system  $H$ . This is completely analogous to the induced-norm of a matrix. Recall that

$$\|A\|_{p-ind} = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|}.$$

The induced norm of a system is

$$\|H\|_{p-ind} = \sup_{u \neq 0} \frac{\|y\|}{\|u\|} = \sup_{u \neq 0} \frac{\|H(u)\|}{\|u\|}.$$

## Fourier Transform

In general, we can define the Fourier integral for vectors of signals  $f(t) = [f_1(t), f_2(t), \dots, f_n(t)]^T$ . However, for better insight, we shall assume in the discussion that  $f$  is a scalar signal. Extensions to signal vectors can be readily made.

**Fourier Integral.** Let  $f : \mathbb{R} \rightarrow \mathbb{C}$  be an integrable function (i.e.  $f \in \mathcal{L}_1^1$ ). The Fourier transform of  $f$  is the function:

$$F(j\omega) = \int_{-\infty}^{+\infty} f(t)e^{-j\omega t} dt.$$

This function is well-defined because we assumed that  $f \in \mathcal{L}_1$

$$|F(j\omega)| = \left| \int_{-\infty}^{+\infty} f(t)e^{-j\omega t} dt \right| \leq \int_{-\infty}^{+\infty} |f(t)| dt < \infty.$$

In general, we expect that  $F(w)$  may not be defined in the above sense for elements of  $\mathcal{L}_2$  since these do not necessarily belong to  $\mathcal{L}_1$ . Still, it can be shown that there is a unique extension of the fourier transform from the set of functions on which it is well-defined to  $\mathcal{L}_2$ . Alternatively, the Fourier integral can be appropriately defined as for functions  $f \in \mathcal{L}_2$ ,

$$\lim_{M \rightarrow \infty} \int_{-M}^M f(t)e^{-j\omega t} dt.$$

which can be shown to exist in the sense of  $\mathcal{L}_2$  and is equal to  $F(j\omega)$  (see [1]).

Finally, it can be shown that the mapping  $f \rightarrow F$  is bijective. That is, the inverse of the fourier transform exists and is unique. Moreover,

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(j\omega)e^{j\omega t} d\omega.$$

**Parseval's Relation.** If  $f(t)$  and  $F(j\omega)$  are a Fourier transform pair, then,  $\|F\|_2^2 = 2\pi\|f\|_2^2$ , i.e.,

$$\int_{-\infty}^{+\infty} \|f(t)\|_2^2 dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \|F(j\omega)\|_2^2 d\omega.$$

Proof. Taking complex conjugates transposes of both sides of

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(j\omega) e^{j\omega t} d\omega,$$

we obtain,

$$(f(t))' = \frac{1}{2\pi} \int_{-\infty}^{+\infty} (F(j\omega))' e^{-j\omega t} d\omega.$$

Now,

$$\begin{aligned} \int_{-\infty}^{+\infty} \|f(t)\|_2^2 dt &= \int_{-\infty}^{+\infty} (f(t))' f(t) dt \\ &= \int_{-\infty}^{+\infty} \left[ \frac{1}{2\pi} \int_{-\infty}^{+\infty} (F(j\omega))' e^{-j\omega t} d\omega \right] f(t) dt \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} (F(j\omega))' \left[ \int_{-\infty}^{+\infty} e^{-j\omega t} f(t) dt \right] d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} (F(j\omega))' F(j\omega) d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \|F(j\omega)\|_2^2 d\omega \end{aligned}$$

**Discrete Time Fourier Transform.** Finally, the fourier transform for a discrete time signal,  $f[n]$ , is defined as

$$F(e^{j\omega}) = \sum_{-\infty}^{+\infty} f[n] e^{-j\omega n},$$

and,

$$f[n] = \frac{1}{2\pi} \int_{2\pi} F(e^{j\omega}) e^{j\omega n} d\omega.$$

Parseval's relation for DT signals is similar to that for CT signals:

$$\sum_{-\infty}^{+\infty} \|f[n]\|_2^2 = \frac{1}{2\pi} \int_{2\pi} \|F(e^{j\omega})\|_2^2 d\omega.$$

## References

- [1] Adams, M., and Guillemin, V. "Measure Theory and Probability," *Birkhauser*: 1996.
- [2] Oppenheim, A., Willsky, A., and Nawab, H. "Signals and Systems," *Prentice Hall*: 1997.