Exercises

Exercise 25.2 **(a)** Find a third-order state-space realization in controller canonical form for the transfer function $H_1(s) = (s+f)/(s+4)^3$, where f is a parameter. (To do this, assume the "A" and "b" of the state-space model are in controller form, then find what "c" and "d" need to be to make the transfer function come out right.) For what values of f does your model lose (i) observability? (ii) controllability?

Similarly, find a first-order controller canonical form realization of the transfer function $H_2(s) = 1/(s-2)$.

(b) Now suppose the realizations in (a) are connected in cascade, with the output of the first system used as the input to the second. The input to the first system then becomes the overall system input, and the output of the second system becomes the overall system output:

$$u \longrightarrow H_1(s) \longrightarrow H_2(s) \longrightarrow y$$

Write down a fourth-order state-space description of the cascade. Is the cascaded system asymptotically stable? — and does your answer depend on f?

Now determine for what values of f the cascaded system loses (i) observability, (ii) controllability. Interpret your results in terms of pole-zero cancellations between $H_1(s)$ and $H_2(s)$. Is there a value of f for which the cascaded system is bounded-input/bounded-output (BIBO) stable but *not* asymptotically stable.

Exercise 25.5 **(a)** Obtain a minimal realization of the system:

$$H(s) = \begin{bmatrix} \frac{s}{(s-1)^2} & \frac{1}{(s-1)} \\ \frac{-6}{(s-1)(s+3)} & \frac{1}{(s+3)} \end{bmatrix}.$$

Explicitly verify its minimality.

(b) Compute the poles (including multiplicities) of this transfer function using the minimal realization you obtained.

Exercise 28.1 Let $(A, B, C, 0)$ be a reachable and observable LTI state-space description of a discrete-time or continuous-time system. Let its input u be related to its output y by the following *output feedback* law:

$$u = Fy + r$$

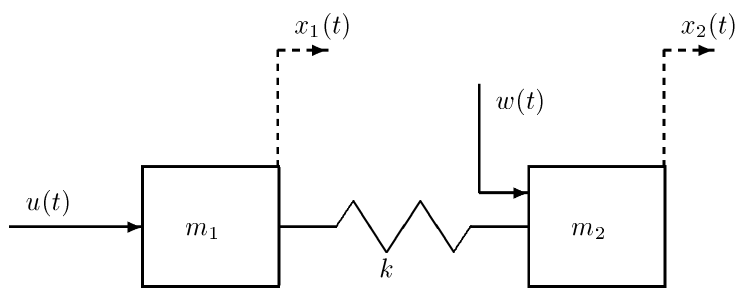
for some constant matrix F, where r is a new external input to the closed-loop system that results from the output feedback.

(a) Write down the state-space description of the system mapping r to y.

(b) Is the new system reachable? Prove reachability, or show a counterexample.

(c) Is the new system observable? Prove observability, or show a counterexample.

Exercise 29.1 Consider the mass-spring system shown in the figure below.



Let $x_1(t)$ denote the position of mass m_1, $x_2(t)$ the position of mass m_2, $x_3(t)$ the velocity of mass m_1, $x_4(t)$ the velocity of mass m_2, $u(t)$ the applied force acting on mass m_1, and $w(t)$ a disturbance force acting on mass m_2, k is the spring constant. There is no damping in the system.

The equations of motion are as follows:

$$\begin{aligned}
\dot{x}_1(t) &= x_3(t) \\
\dot{x}_2(t) &= x_4(t) \\
\dot{x}_3(t) &= -(k/m_1)x_1(t) + (k/m_1)x_2(t) + (1/m_1)u(t) \\
\dot{x}_4(t) &= (k/m_2)x_1(t) - (k/m_2)x_2(t) + (1/m_2)w(t)
\end{aligned}$$

The output is simply the position of mass m_2, so

$$y(t) = x_2(t)$$

Assume the following values for the parameters:

$$m_1 = m_2 = 1; \quad k = 1$$

(a) Determine the natural frequencies of the system, the zeros of the transfer function from u to y, and the zeros of the transfer function from w to y.

(b) Design an observed-based compensator that uses a feedback control of the form $u(t) = F\hat{x}(t)+r(t)$, where $\hat{x}(t)$ is the state-estimate provided by an observer. Choose F such that the poles of the transfer function from r to y are all at -1. Design your observer such that the natural frequencies governing observer error decay are all at -5.

(c) Determine the closed-loop transfer function from the disturbance w to the output y and obtain its Bode magnitude plot. Comment on the disturbance rejection properties of your design.

(d) Plot the transient response of the two position variables and of the control when $x_2(0) = 1$ and all the other state variables, including the compensator state variables, are initially zero.

(e) Plot the transient response of the two position variables and of the control when the system is initially at rest and the disturbance $w(t)$ is a unit step at time $t = 0$.

Exercise 29.2 Reduced Order Observer

The model-based observer that we discussed in class always has dimension equal to the dimension of the plant. Since the output measures part of the states (or linear combinations), it seems natural that only a subset of the states need to be estimated through the observer. This problem shows how one can derive a reduced order observer.

Consider the following dynamic system with states $x_1 \in \mathbb{R}^r$, $x_2 \in \mathbb{R}^p$:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} u,$$

and

$$y = x_2.$$

Since x_2 is completely available, the reduced order observer should provide estimates only for x_1, and its dimension is equal to r, the dimension of x_1. Thus

$$\hat{x} = \begin{pmatrix} \hat{x}_1 \\ x_2 \end{pmatrix}.$$

One may start with the following potential observer:

$$\dot{\hat{x}}_1 = A_{11}\hat{x}_1 + A_{12}y + B_1 u + L(y - \hat{y})$$

Since $\hat{y} = C\hat{x} = x_2$ (since x_2 is known exactly), the correction term in the above equation is equal to zero ($L(y - \hat{y}) = 0$). This indicates that this proceedure may not work.

Suppose instead, that we define a new variable $z = x_1 - Lx_2$, where L is an $r \times p$ matrix that we will choose later. Then if we can derive an estimate for z, denoted by $\hat{z}$, we immediately have an estimate for x_1, namely, $\hat{x}_1 = \hat{z} + Lx_2$.

(a) Express $\dot{z}$ in terms of z, y, and u. Show that the state matrix (matrix multiplying z) is given by $A_{11} - LA_{21}$.

(b) To be able to place the poles of $A_{11} - LA_{21}$ in the left half plane, the pair (A_{11}, A_{21}) should be observable (i.e., a system with dynamic matrix A_{11} and output matrix A_{21} should be observable). Show that this is the case if and only if the original system is observable.

(c) Suggest an observer for z. Verify that your choice is good.

(d) Suppose a constant state feedback matrix F has been found such that $A + BF$ is stable. Since not all the states are availabe, the control law can be implemented as:

$$u = F\hat{x} = F_1\hat{x}_1 + F_2 x_2$$

where $F = \begin{pmatrix} F_1 & F_2 \end{pmatrix}$ is decomposed conformally with x_1 and x_2. Where do the closed loop poles lie? Justify your answer.

Exercise 29.5 Consider a plant described by the transfer function matrix

$$P(s) = \begin{pmatrix} \frac{1}{s-1} & \frac{1}{s-1} \\ \frac{2s-1}{s(s-1)} & \frac{1}{s-1} \end{pmatrix}$$

(a) Design a model-based (i.e. observer-based) controller such that the closed loop system has all eigenvalues at $s = -1$.

(b) Suppose that $P_{11}(s)$ is perturbed to $\frac{1+\epsilon}{s-1}$. For the controller you designed, give the range of ϵ for which the system remains stable. Discuss your answer.