MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science

6.241: Dynamic Systems—Fall 2003

RECITATION 9

Stability Robustness[1]

We have seen in a previous lectures that given a plant, P, we can design a controller, K, in feedback, such that the closed-loop system is stable and meets certain performance criteria such as disturbance or noise rejection. We may design a controller assuming some plant model, P, but, if there were any uncertainties in our P, such as unmodelled dynamics that may be excited by the controller, certain nonlinearities in the model that were not considered in the design of the controller, uncertainty in the values of the plant parameters, then the controller K, may not stabilize the plant (see example 19.3). So, the question is, how do we design our controller such that it stabilizes our plant (and delivers certain performance specifications) in the presence of plant uncertainties. This is the question of robust stabilization (or performance). We begin by characterizing the uncertainty in P.

Plant Uncertainty

When modelling the plant, there may have been high-frequency dynamics that were not modeled, nonlinearities that were neglected, or unknown paramaters whose values were assumed. The real or accurate plant model may actually be one of several representations, in some set Π. The idea is to imbed this set Π in some larger set, Ω, which has a certain structure. Such Ω sets include the following:

(i)	Additve Uncertainty	$\Omega = \{P(s) \mid P(s) = P_o(s) + W(s)\Delta(s)\}$	(Figure 1)
(ii)	Multiplicative Uncertainty	$\Omega = \{P(s) \mid P(s) = P_o(s)(I + W(s)\Delta(s))\}$	
		$\Omega = \{P(s) \mid P(s) = (I + W(s)\Delta(s))P_o(s)\}$	(Figure 2)
(iii)	Feedback Uncertainty	$\Omega = \left\{P(s) \mid P(s) = (I - W(s)\Delta(s))^{-1}P_o(s)\right\}$	(Figure 3)

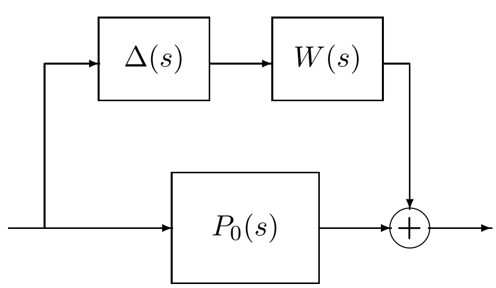

Figure 1: Block Diagram for additive representation of uncertainty.

where we assume the following,

[1]Based on Keith Santarelli's recitation 11/16/2001.

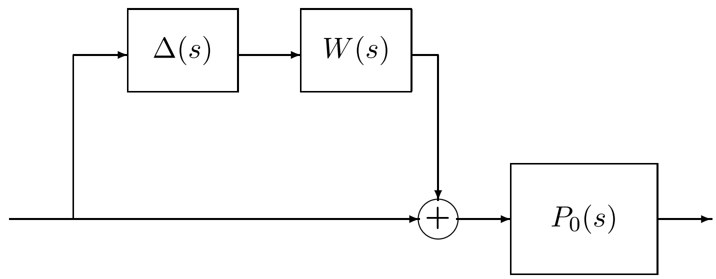

Figure 2: Block Diagram for multiplicative representation of uncertainty.

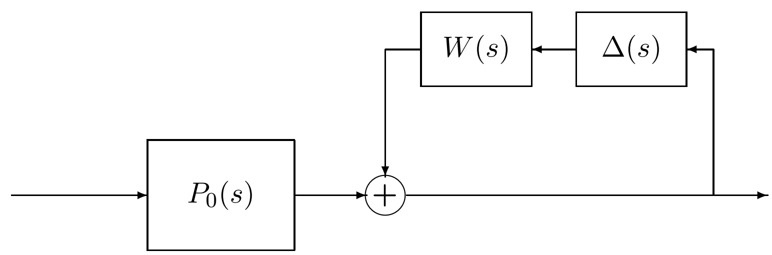

Figure 3: Block Diagram for feedback representation of uncertainty.

(1) All unstable pole locations of the actual plant are known.

(2) $W(s)$ is a stable weighting matrix that reflects information known about the uncertainty. For example, the additive uncertainty model is appropriate when $P_o(s)$ accurately represents the plant at low frequencies, but does a poor job at high frequencies. In this case, $W(j\omega)$ should capture the information about the frequency content of the uncertainty, because $P - P_o = W\Delta$, and we want $|P(j\omega) - P_o(j\omega)|$ to be small for low frequencies and allow it to be large but bounded at high frequencies. $W(s)$ that reflects this is a high pass filter.

(3) $\Delta(s)$ is a stable uncertainty matrix that, by proper choice of $W(s)$, can be normalized and assumed to have $\|\Delta\|_\infty < 1$. If Δ were 1×1, the shape of $|\Delta(j\omega)|$ and $\angle\Delta(j\omega)$ can be anything.

Depending on the kind of uncertainty in the plant, one of the above representations of uncertainty, (i), (ii), or (iii), may be more appropriate than the others.

Linear Fractional Transformation

Suppose that we built some feedback configuration around our uncertain plant, $P(s)$. One possible configuration is the standard servo configuration shown below (Figure 4). This system can be transformed into the equivalent representation of figure 5, where

- In genral, r and y can be any physically accessible signals of interest,
- We have isolated Δ from the rest of the systen and lumped everything into G, the transfer

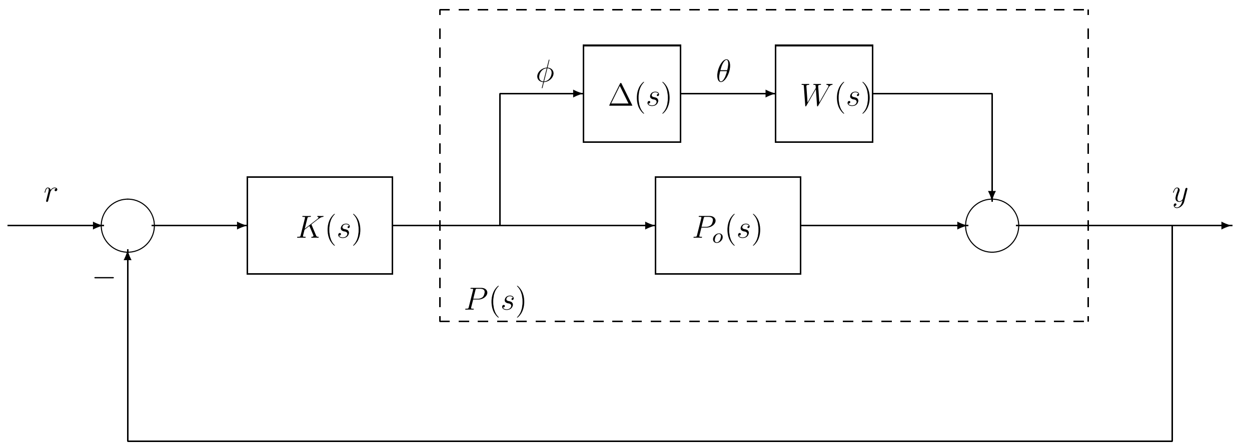

Figure 4: Standard servo configuration around a plant with additive uncertainty.

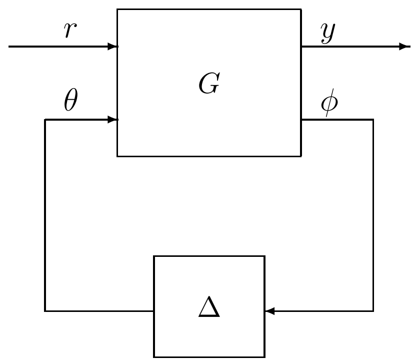

Figure 5: Linear fractional transformation.

function from $\begin{bmatrix} \theta & r \end{bmatrix}^T$ to $\begin{bmatrix} \phi & y \end{bmatrix}^T$ in the absence of Δ:

$$\begin{bmatrix} \phi \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} M(s) & N(s) \\ J(s) & L(s) \end{bmatrix}}_{G(s)} \begin{bmatrix} \theta \\ r \end{bmatrix}. \tag{1}$$

Here, $G(s)$ will depend on the structure of $P(s)$ and the feedback configuration. For example, if $P(s)$ has the structure of the additive uncertainty model with a controller in the servo feedback configuration, shown in figure 4, the transfer function from θ to ϕ, $M(s)$ would be $M(s) = -(I + KP_o)^{-1}KW$. If P had uncertainty of the multiplicative structure, as shown in figure 6, then $M(s) = -(I + P_oK)^{-1}P_oKW$.

Now, substituting $\theta = \Delta\phi$ in equation 1, we have

$$\begin{aligned} y &= J\theta + Lr = J\Delta\phi + Lr \\ \phi &= M\theta + Nr = M\Delta\phi + Nr \end{aligned}$$

So, $\phi = (I - M\Delta)^{-1}Nr$, and $y = (L + J\Delta(I - M\Delta)^{-1}N)r$ i.e., the closed loop transfer function is $L + J\Delta(I - M\Delta)^{-1}N$.

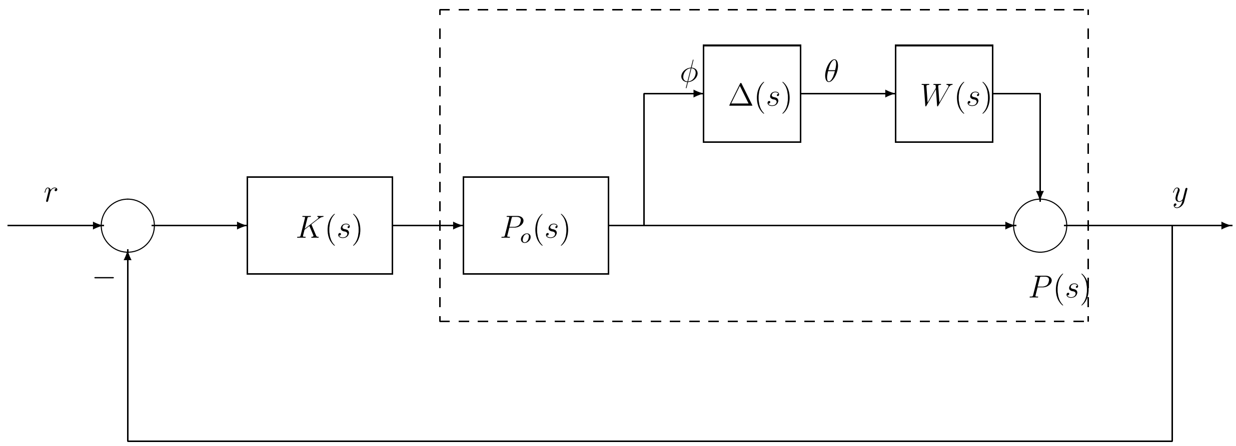

Figure 6: Standard servo configuration around a plant with multiplicative uncertainty.

Small Gain Theorem

For robust stability, we require that the controller that we design, K, result in a stable closed loop system for all possible P in Ω, i.e. all allowable Δ. Thus, K should stabilize the nominal plant P_o; hence, $L(s)$, $J(s)$, and $N(s)$ are all stable (you van verify this by writing $L(s)$, $J(s)$, and $N(s)$ for some feedback and plant uncertainty structure and noting that $(I + P_o K)^{-1}$, K, and W are stable by assumption). So, the requirement that the closed loop system is stable for all admissible Δ boils down to requiring that $(I - M\Delta)^{-1}$ be stable for all admissible Δ. Stability is guranteed by the small gain theorem.

Small Gain Theorem. Assume there is no structure resrtiction on Δ, suppose $\|M\|_\infty \leq \gamma_M$ and $\|\Delta\|_\infty \leq \gamma_\Delta$, then $(I - M\Delta)^{-1}$ will have finite L_2 gain, i.e., $\|(I - M\Delta)^{-1}\|_\infty \leq \gamma$ if and only if $\gamma_M \gamma_\Delta < 1$.

The above statement of the small gain theorem holds for Δ nonlinear or time-varying.

Example 1: Suppose that $P(s) = (1 + W(s)\Delta(s))P_o(s)$ where $P_o = \frac{1}{s-1}$ and $W(s) = \frac{2}{s+10}$ and $\Delta(s)$ is an arbitrary stable LTI system with $\|\Delta\|_\infty \leq 2$. We wish to design a controller $K(s) = k$, connected in the standard servo configuration, such that the closed loop system is stable. The question is, for which range of k will the closed loop system be guaranteed to be stable.

First, we calculate $M(s)$. Recall that for this configuration and plant model, $M(s) = -(I + P_o K)^{-1} P_o K W$. So,

$$M(s) = \frac{-P_o K W}{1 + P_o K} = \frac{-2k}{(s+10)(s-1+k)}.$$

We now apply the small gain theorem. First, note that in order to do that, we need $M(s)$ to be stable. So this condition immediately gives us that $k > 1$. Now, by the small gain theorem, we need:

$$|M(j\omega)| < \frac{1}{2} \quad \forall\omega,$$

where,

$$|M(j\omega)| = \sqrt{\frac{4k^2}{(\omega^2+10)(\omega^2+(k-1)^2)}}$$

and

$$\max_{\omega} |M(j\omega)| = \left|\frac{2k}{10(k-1)}\right|.$$

So, $\left|\frac{2k}{10(k-1)}\right| < \frac{1}{2}$ and $k > 1$ imples that $k > \frac{5}{3}$.

Example 2: Suppose the plant is a low pass filter with a pole very close to the origin, $P(s) = \frac{1}{s+\epsilon}$ where $\epsilon > 0$ is a small number. We wish to design a feedback system with a constant gain controller, $K(s) = k$, to move the pole away from the imaginary axis, if possible. The constraint on the controller is that is saturates if the input has a magnitude greater than 1:

$$y = \mathrm{sat}(x) = \begin{cases} x & |x| < 1 \\ 1 & x > 1 \\ -1 & x < -1 \end{cases}$$

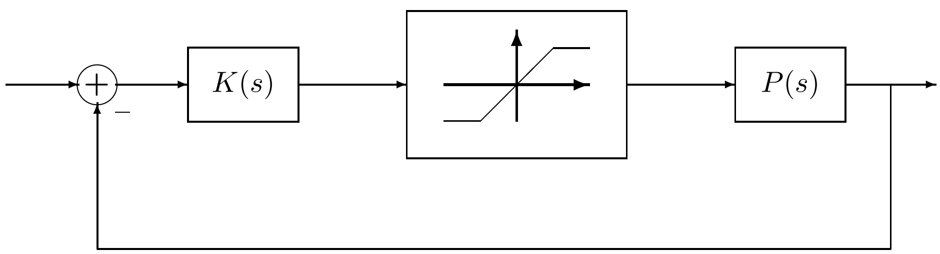

Figure 7: Feedback system model.

Our approach will be to describe the saturation block as an uncertainty block and use the small gain theorem to obtain conditions on k that will guarantee stability of the closed loop system in the presence of Δ.

Suppose we replace the saturation block by Δ with $\|\Delta\|_\infty \leq 1$. The saturation block belongs to this class of systems because its L_2 gain is less than or equal to one (the output signal of the saturation element will have energy less than or equal to the energy of the input signal). The uncertainty model is shown in figure 8. Before we go further, we need to make sure that the uncertainty model we chose makes sense. A little thought leads us to the conclusion that our choice of representing of the saturation block as an uncertainty element Δ is a poor one. First of all, if $\Delta = 0$ then we have an open loop system. This does not reflect the reality of the system, namely, that when the loop signals are small, the saturation block has no effect (i.e. no uncertainty), and the closed loop system is linear. Second, by allowing Δ to be anything as long as $\|\Delta\|_\infty \leq 1$, our uncertainty model does not capture any information that we do have about the uncertainty, namely, that is is a saturation block.

Consider the following model for the saturation block, (figure 9), where

$$\theta = \Delta(\phi) = \begin{cases} 1-\phi & \phi < -1 \\ 0 & |\phi| < 1 \\ -1-\phi & \phi > -1 \end{cases}.$$

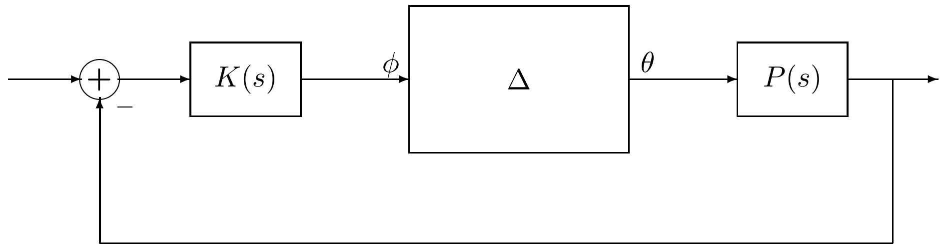

Figure 8: "Uncertainty" model for feedback system.

This choice of Δ is depicted graphically in figure 10. Note that

$$y = \phi + \theta = x + \Delta(x) = x + \left\{ \begin{array}{rl} 1 - x & x < -1 \\ 0 & |x| < 1 \\ -1 - x & x > -1 \end{array} \right\} = \mathrm{sat}(x),$$

as we require. Furthermore, $\|\Delta(\phi)\|_\infty \leq 1$. So, we apply the small gain theorem, with $M(s) =$

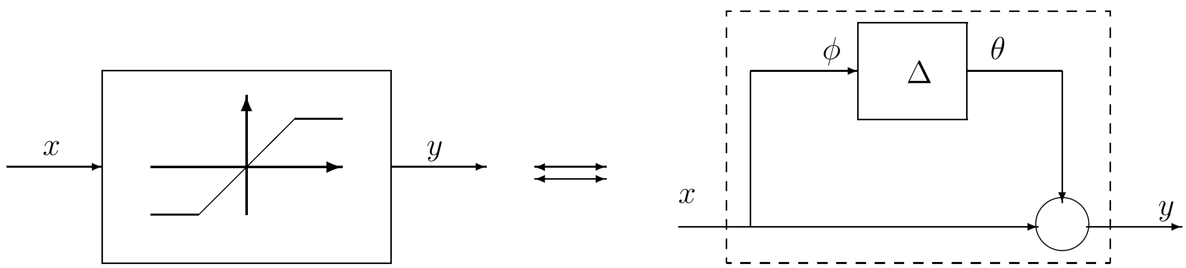

Figure 9: "Uncertainty" model for the saturation block.

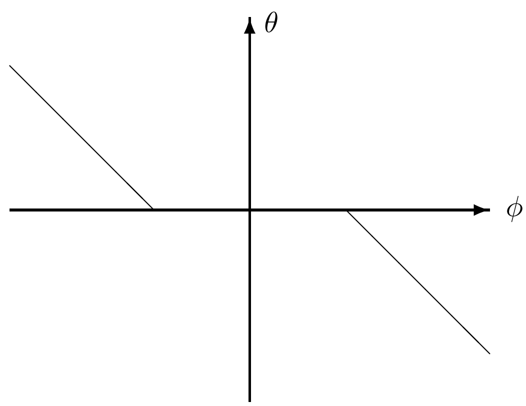

Figure 10: Choice of Δ which yields saturation for better "uncertainty" model.

$\frac{-k}{s+k+\epsilon}$. So, $\|M(s)\|_\infty = \frac{k}{k+\epsilon}$ which, for $k > 0$ and $\epsilon > 0$ is always less than one, for any k. Thus, we can push the pole of the plant as far into the left half plane as we like, and the closed loop system will remain stable.

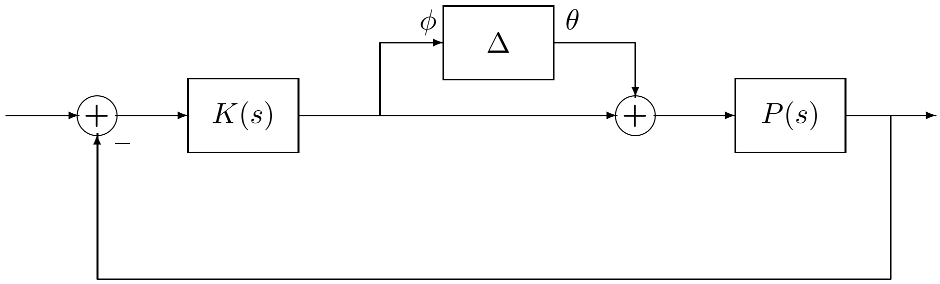

Figure 11: Better "uncertainty" model for feedback system.

References

[1] Zhou, K., with Doyle, J. "Essentials of Robust Control." *Prentice Hall*: 1998.