Lectures on Dynamic Systems and Control

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Chapter 22

Reachability of DT LTI Systems

22.1 Introduction

We now begin a series of lectures to address the question of synthesizing feedback controllers. This objective requires a detailed understanding of how inputs impact the states of a given system, a notion we term *reachability*. Also, this objective requires a detailed understanding of the information the output provides about the rest of the states of the dynamic system, a notion we term observability. These notions together define the minimal set of conditions under which a stabilizing feedback controller exists.

22.2 The Reachability Problem

In previous lectures we have examined solutions of state-space models, the stability of undriven models, some properties of interconnections, and input-output stability. We now turn to a more detailed examination of how inputs affect states, for the n^{th}-order DT system

$$x(i+1) = Ax(i) + Bu(i) \; . \tag{22.1}$$

(The discussion of reachability in the DT case is generally simpler than in the CT case that we will consider next Chapter, but some structural subtleties that are hidden in the CT case become more apparent in the DT case. For the most part, however, DT results parallel CT results quite closely.) Recall that

$$\begin{aligned} x(k) &= A^k x(0) + \sum_{i=0}^{k-1} A^{k-i-1} B u(i) \\ &= A^k x(0) + \left[A^{k-1}B \mid A^{k-2}B \mid \cdots \mid B \right] \begin{pmatrix} u(0) \\ u(1) \\ \vdots \\ u(k-1) \end{pmatrix} \\ &= A^k x(0) + R_k \, \mathcal{U}_k \end{aligned} \tag{22.2}$$

where the definition of R_k and $\mathcal{U}_k$ should be clear from the equation that precedes them. Now consider whether and how we may choose the input sequence $u(i)$, $i \in [0,\ k-1]$, so as to move the system from $x(0) = 0$ to a desired target state $x(k) = d$ at a given time k. If there is such an input, we say that the state d is **reachable** in k steps. It is evident from (22.2) that — assuming there are no constraints placed on the input — the set $\mathbb{R}_k$ of states reachable from the origin in k steps, or the *k-reachable set*, is precisely the range of R_k, i.e.

$$\mathbb{R}_k = Ra(R_k) \tag{22.3}$$

The k-reachable set is therefore a *subspace*, and may be referred to as the k-reachable subspace. We call the matrix R_k the *k-step reachability matrix.*

Theorem 22.1

For $k \leq n \leq \ell$,

$$Ra(R_k) \subseteq Ra(R_n) = Ra(R_\ell) \tag{22.4}$$

so the set of states reachable from the origin in some (finite) number of steps by appropriate choice of control is precisely the subspace of states reachable in n steps.

Proof.
The fact that $Ra(R_k) \subseteq Ra(R_n)$ for $k \leq n$ follows trivially from the fact that the columns of R_k are included among those of R_n. To show that $Ra(R_n) = Ra(R_\ell)$ for $\ell \geq n$, note from the Cayley-Hamilton theorem that A^i for $i \geq n$ can be written as a linear combination of $A^{n-1}, \cdots, A, I$, so all the columns of R_ℓ for $\ell \geq n$ are linear combinations of the columns of R_n. Thus (22.4) is proved, and the rest of the statement of the theorem follows directly.

In view of Theorem 22.1, the subspace of states reachable in n steps, *i.e.* $Ra(R_n)$, is referred to as *the* reachable subspace, and will be denoted simply by $\mathbb{R}$; any reachable target state, *i.e.* any state in $\mathbb{R}$, is reachable in n steps (or less). The system is termed a *reachable system* if all of $\mathbb{R}^n$ is reachable, *i.e.* if $\text{rank}(R_n) = n$. The matrix

$$R_n = \left[\, A^{n-1}B \mid A^{n-2}B \mid \ \cdots \ \mid B \,\right], \tag{22.5}$$

is termed the *reachability matrix* (often written with its block entries ordered oppositely to the order that we have used here, but this is not significant).

Example 22.1 Consider the single-input system

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k).$$

The reachable subspace is evidently (from symmetry) the line $x_1 = x_2$. This system is not reachable.

The following alternative characterization of $\mathbb{R}_k$ is useful, particularly because its CT version will play an important role in our development of the CT reachability story. Let us first define the *k-step reachability Gramian* $\mathcal{P}_k$ by

$$\mathcal{P}_k = R_k R_k^T = \sum_{i=0}^{k-1} A^i B B^T (A^T)^i \tag{22.6}$$

This matrix is therefore symmetric and positive semi-definite. We then have the following result.

Lemma 22.1

$$Ra(\mathcal{P}_k) = Ra(R_k) = \mathbb{R}_k \ . \tag{22.7}$$

Proof.
It is easy to see that $Ra(\mathcal{P}_k) \subset Ra(R_k)$. For the reverse inclusion, we can equivalently show that

$$Ra^{\perp}(\mathcal{P}_k) \subset Ra^{\perp}(R_k)$$

For this, note that

$$\begin{aligned} q^T\mathcal{P}_k = 0 \quad &\Longrightarrow \quad q^T\mathcal{P}_k q = 0 \\ &\Longleftrightarrow \quad \langle R_k^T q, R_k^T q\rangle = 0 \\ &\Longleftrightarrow \quad q^T R_k = 0 \end{aligned}$$

so any vector in $Ra^{\perp}(\mathcal{P}_k)$ is also in $Ra^{\perp}(R_k)$.
Thus the reachable subspace can equivalently be computed as $Ra(P_\ell)$ for any $\ell \geq n$. If the system is *stable*, then $P_\infty := P$ is well defined, and is easily shown to satisfy the Lyapunov equation

$$APA^T - P = -BB^T \tag{22.8}$$

We leave you to show that (22.8) has a (unique) positive definite (and hence full rank) solution P if and only if the system (A, B) is reachable.

Reachability from an Arbitrary Initial State

Note from (22.2) that getting from a nonzero starting state $x(0) = s$ to a target state $x(k) = d$ requires us to find a $\mathcal{U}_k$ for which

$$d - A^k s = R_k\,\mathcal{U}_k \tag{22.9}$$

For arbitrary d, s, the requisite condition is the same as that for reachability from the origin. Thus we can get from an arbitrary initial state to an arbitrary final state if and only if the system is reachable (from the origin); and we can make the transition in n steps or less, when the transition is possible.

Controllability versus Reachability

Now consider what is called the **controllability** problem, namely that of bringing an arbitrary initial state $x(0)$ to the origin in a finite number of steps. From (22.2) we see that this requires solving

$$-A^k x(0) = R_k\,\mathcal{U}_k \tag{22.10}$$

If A is invertible and $x(0)$ is arbitrary, then the left side of (22.10) is arbitrary, so the condition for controllability of $x(0)$ to the origin in a finite number of steps is precisely that $\text{rank}(R_k) = n$ for some k, *i.e.* just the reachability condition that $\text{rank}(R_n) = n$.

If, on the other hand, A is singular (*i.e.* has eigenvalues at 0), then the left side of (22.10) will be confined to a subspace of the state space, even when $x(0)$ is unrestricted. The range of A^k for a singular A may decrease initially, but $Ra(A^k) = Ra(A^n)$ for $k \geq n$ (since by stage n the Jordan blocks associated with the zero eigenvalues of A are all guaranteed to have been "zeroed out" in A^n). Meanwhile, as we have seen, the range of R_k may increase initially, but $Ra(R_k) = Ra(R_n)$ for $k \geq n$.

It follows from these facts and (22.10) that an arbitrary initial state is controllable to 0 in finite time, *i.e.* the system is controllable, iff

$$Ra(A^n) \subset Ra(R_n) \tag{22.11}$$

For invertible A, we recover our earlier condition. (The distinction between reachability and controllability is not seen in the CT case, because the state transition matrix there is e^{At} rather than A^k, and is always invertible.)

22.3 Modal Aspects

The following result begins to make the connection of reachability with modal structure.

Corollary 22.1

The reachable subspace $\mathbb{R}$ is A-invariant, *i.e.* $x \in \mathbb{R} \Longrightarrow Ax \in \mathbb{R}$. We write this as $A\mathbb{R} \subset \mathbb{R}$
Proof.
We first show

$$Ra(AR_n) \subset Ra(R_n) \tag{22.12}$$

For this, note that

$$AR_n = \left[\begin{array}{c|c|c|c} A^nB & A^{n-1}B & \cdots & AB \end{array}\right]$$

The last $n-1$ blocks are present in R_n, while the Cayley-Hamilton theorem allows us to write A^nB as a linear combination of blocks in R_n. This establishes (22.12). It follows that $x = R_n\alpha \Longrightarrow Ax = AR_n\alpha = R_n\beta \in \mathbb{R}$.

Some feel for how this result connects to modal structure may be obtained by considering what happens if the subspace $\mathbb{R}$ is one-dimensional. If v $(\neq 0)$ is a basis vector for $\mathbb{R}$, then Corollary 22.1 states that

$$Av = \lambda v \tag{22.13}$$

for some λ, *i.e.* $\mathbb{R}$ is the space spanned by an *eigenvector* of A. More generally, it is true that any A-invariant subspace is the span of some eigenvectors and generalized eigenvectors of A. (It turns out that $\mathbb{R}$ is the smallest A-invariant subspace that contains $Ra(B)$, but we shall not pursue this fact.)

Standard Form for Unreachable Systems

If a system of the form (22.1) is unreachable, it is convenient to choose coordinates that highlight this fact. Specifically, we shall show how to change coordinates (using a similarity transformation) from $x = Tz$ to

$$z = T^{-1}x = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

where z_1 is an r-vector and z_2 is an $(n-r)$-vector, with r denoting the dimension of the reachable subspace, $r = \dim \mathbb{R}$. In these new coordinates, the system (22.1) will take the form

$$\begin{bmatrix} z_1(k+1) \\ z_2(k+1) \end{bmatrix} = \left[\begin{array}{c|c} A_1 & A_{12} \\ \hline 0 & A_2 \end{array}\right] \begin{bmatrix} z_1(k) \\ z_2(k) \end{bmatrix} + \left[\begin{array}{c} B_1 \\ \hline 0 \end{array}\right] u(k) \tag{22.14}$$

with the reachable subspace being the subspace with $z_2 = 0$. We shall refer to a system in the form (22.14) as being in the *standard form* for an unreachable system.

The matrix T is constructed as follows. Let $T_1^{n\times r}$ be a matrix whose columns form a basis for the reachable subspace, *i.e.*

$$\mathcal{R}a(T_1) = \mathcal{R}a(R_n) \ ,$$

and let $T_2^{n\times(n-r)}$ be a matrix whose columns are independent of each other and of those in T_1. Then choose

$$T = [\, T_1 \,|\, T_2 \,] \ .$$

This matrix is invertible, since its columns are independent by construction. We now claim that

$$\begin{aligned} A\,[\, T_1 \,|\, T_2 \,] \ &= \ T\bar{A} \ = [\, T_1 \,|\, T_2 \,] \begin{bmatrix} A_1^{r\times r} & A_{12} \\ 0 & A_2 \end{bmatrix} \\ B \ &= \ T\bar{B} \ = [\, T_1 \,|\, T_2 \,] \begin{bmatrix} B_1^{r\times m} \\ --- \\ 0 \end{bmatrix} . \end{aligned} \tag{22.15}$$

Our reasoning is as follows. Since the reachable subspace is A-invariant, the columns of AT_1 must remain in $\mathcal{R}a(T_1)$, which forces the 0 block in the indicated position in $\bar{A}$. Similarly, the presence of the zero block in $\bar{B}$ is a consequence of the fact that the columns of B are in the reachable subspace.

The above standard form is not uniquely defined, but it can be shown (we leave you to show it!) that any two such standard forms are related by a block upper triangular similarity transformation. As a result, A_1 and A_2 are *unique up to similarity transformations* (so, in particular, their Jordan forms are uniquely determined).

From (22.14) it is evident that if $z_2(0) = 0$ then the motion of $z_1(k)$ is described by the r^{th}-order *reachable* state-space model

$$z_1(k+1) = A_1 z_1(k) + B_1 u(k). \tag{22.16}$$

This is also called the *reachable subsystem* of (22.1) or (22.14). The eigenvalues of A_1, which we may refer to as the *reachable eigenvalues*, govern the ZIR in the reachable subspace. Also, the behavior of $z_2(k)$ is described by the *undriven* state-space model

$$z_2(k+1) = A_2 z_2(k) \tag{22.17}$$

and is governed by the eigenvalues of A_2, which we may call the *unreachable* eigenvalues.

There is no loss of generality in assuming a given unreachable system has been put in the standard form for unreachable systems; proofs of statements about unreachable systems are often much more transparent if done in these coordinates.

Modal Reachability Tests

An immediate application of the standard form is to prove the following *modal test* for (un)reachability.

Theorem 22.2

The system (22.1) is unreachable if and only if $w^T B = 0$ for some left eigenvector w^T of A. We say that the corresponding eigenvalue λ is an unreachable eigenvalue.

Proof.

If $w^T B = 0$ and $w^T A = \lambda w^T$ with $w^T \neq 0$, then $w^T AB = \lambda w^T B = 0$ and similarly $w^T A^k B = 0$, so $w^T R_n = 0$, *i.e.* the system is unreachable.

Conversely, if the system is unreachable, transform it to the standard form (22.14). Now let w_2^T denote a left eigenvector of A_2, with eigenvalue λ. Then $w^T = [\ 0 \quad w_2^T\ ]$ is a left eigenvector of the transformed A matrix, namely $\bar{A}$, and is orthogonal to the (columns of the) transformed B, namely $\bar{B}$.

An alternative form of this test appears in the following result.

Corollary 22.2

The system (22.1) is unreachable if and only if $[\ zI - A \mid B\ ]$ loses rank for some $z = \lambda$. This λ is then an unreachable eigenvalue.

Proof.

The matrix $[\ zI - A \mid B\ ]$ has less than full rank at $z = \lambda$ iff $w^T[\ sI - A \mid B\ ] = 0$ for some $w^T \neq 0$. But this is equivalent to having a left eigenvector of A being orthogonal to (the columns of) B.

Example 22.2

Consider the system

$$x(k+1) = \underbrace{\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}}_{A} x(k) + \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{B} u(k)$$

Left eigenvectors of A associated with its eigenvalue at $\lambda = 3$ are $w_1^T = [\ 1 \quad 0\ ]$ and $w_=^T[\ 0 \quad 1\ ]$, neither of which is orthogonal to B. However, $w_0^T = [\ 1 \quad -1\ ]$ is *also* a left eigenvector associated with $\lambda = 3$, and *is* orthogonal to B. This example drives home the fact that the modal unreachability test only asks for *some* left eigenvector to be orthogonal to B.

Jordan Chain Interpretation

Recall that the system (22.1) may be thought of as having a collection of "Jordan chains" at its core. Reachability, which we first introduced in terms of reaching target states, turns out to also describe our ability to independently "excite" or drive the Jordan chains. This is the implication of the reachable subspace being an A-invariant subspace, and is the reason why the preceding modal tests for reachability exist.

The critical thing for reachability is to be able to excite the *beginning* of each chain; this excitation can then propagate down the chain. An additional condition is needed if several chains have the same eigenvalue; in this case, we need to be able to *independently* excite the beginning of each of these chains. (Example 22.2 illustrates that reachability is lost otherwise; with just a single input, we are unable to excite the two identical chains independently.) With distinct eigenvalues, we do not need to impose this independence condition; the distinctness of the eigenvalues permits independent motions.

Some additional insight is obtained by considering the distinct eigenvalue case in more detail. In this case, A in (22.1) is diagonalizable, and $A = V\Lambda W$, where the columns of V are the right eigenvectors of A and the rows of W are the left eigenvectors of A. For $x(0) = 0$ we have

$$x(k) = \sum_{\ell=1}^{n} v_\ell w_\ell^T B g_\ell(k) \tag{22.18}$$

where

$$g_\ell(k) = \sum_{i=0}^{k-1} \lambda_\ell^{k-i-1} u(i) \tag{22.19}$$

If $w_j^T B = 0$ for some j, then (22.18) shows that $x(k)$ is confined to the span of $\{v_\ell\}_{\ell \neq j}$, *i.e.* the system is not reachable. For example, suppose we have a second-order system ($n = 2$), and suppose $w_1^T B = 0$. Then if $x(0) = 0$, the response to *any* input must lie along v_2. This means that v_2 spans the reachable space, and that any state which has a component along v_1 is not reachable.

Exercises

Exercise 22.1 Suppose you are given the single-input, nth-order system $x(k+1) = Ax(k) + bu(k)$, and assume the control u at every time step is confined to lie in the interval $[0, 1]$. Assume also that an eigenvalue of A, say λ_1, is real and nonnegative. Show that the set of states reachable from the origin is confined to one side of a hyperplane through the origin in $\mathcal{R}^n$. (Hint: An eigenvector associated with λ_1 will help you make the argument.)

[A hyperplane through the origin is an $(n-1)$-dimensional subspace defined as the set of vectors x in $\mathcal{R}^n$ for which $a'x = 0$, where a is some fixed nonzero vector in $\mathcal{R}^n$. Evidently a is normal to the hyperplane. The two "sides" of the hyperplane, or the two "half-spaces" defined by it, are the sets of x for which $a'x \leq 0$ and $a'x \geq 0$.]

Exercise 22.2 Given the system

$$x(k+1) = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} x(k) + \begin{pmatrix} d \\ e \end{pmatrix} u(k)$$

where a, b, c, d, e are scalars, deduce precisely what condition these coefficients satisfy when the system is *not* reachable. Draw a block diagram corresponding to the above system and use it to interpret the following special cases in which reachability is lost: (a) $e = 0$; (b) $b = 0$ and $d = 0$; (c) $b = 0$ and $c = a$.

Exercise 22.3 **(a)** Given m-input system $x(k+1) = Ax(k) + Bu(k)$, where A is the Jordan-form matrix

$$A = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}$$

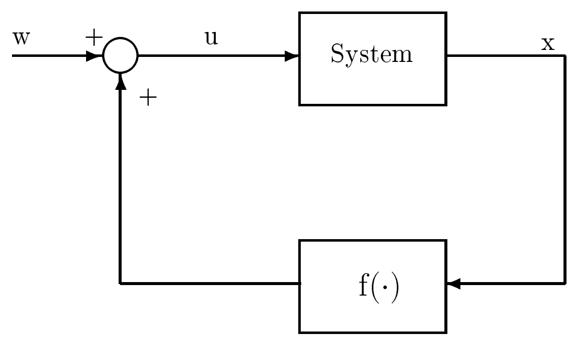
obtain conditions that are necessary and sufficent for the system to be reachable. (Hint: Your conditions should involve the *rows* b_i of B. Some form of the modal reachability test will — not surprisingly! — lead to the simplest solution.)

(b) Generalize this reachability result to the case where A is a general $n \times n$ Jordan-form matrix.

(c) Given the *single-input, reachable* system $x(k+1) = Ax(k) + bu(k)$, show that there can be only *one* Jordan block associated with each distinct eigenvalue of A.

Exercise 22.4 Given the n-dimensional reachable system $x(k+1) = Ax(k) + Bu(k)$, suppose that $u(k)$ is generated according to the nonlinear feedback scheme shown in the figure, where $u(k) = w(k) + f(x(k))$, with $f(.)$ being an arbitrary but known function, and $w(k)$ being the new control input for the closed-loop system.

Show that $w(k)$ can always be chosen to take the system state from the origin to any specified target state in no more than n steps. You will thereby have proved that *reachability is preserved under (even nonlinear) state feedback.*



$$x_{k+1} = Ax_k + B(w_k + f(x_k))$$

Exercise 22.5 Consider the following linear SISO System, Σ:

$$\begin{aligned} x(k+1) &= A(k)x(k) + B(k)u(k) \\ y(k) &= C(k)x(k) + D(k)u(k) \end{aligned}$$

where $A(k) = A(k+N)\, \forall k \geq 0$, similarly for $B(k), C(k)$, and $D(k)$.

(a) Show that Σ is N-Periodic, i.e., for zero initial conditions, show that if y is the output response for some input u, then $y(k-N)$ is the output response for $u(k-N)$. Assume for simplicity that $u(k) = 0$ for $k < 0$.

We want to get a different representation of this system that is easier to work with. To achieve this, we will group together every N successive inputs starting from $k = 0$. We will also do the same for the output. To be more precise, we will define a mapping L, called a *lifting*, such that

$$L : (u(0), u(1), u(2), \ldots, u(k), \ldots) \to \tilde{u}$$

where

$$\tilde{u} = \left(\begin{pmatrix} u(0) \\ u(1) \\ \vdots \\ u(N-1) \end{pmatrix}, \begin{pmatrix} u(N) \\ u(N+1) \\ \vdots \\ u(2N-1) \end{pmatrix}, \ldots, \begin{pmatrix} u(kN) \\ u(kN+1) \\ \vdots \\ u((k+1)N-1) \end{pmatrix}, \ldots \right).$$

Similarly, $L : y \to \tilde{y}$.

(b) Show that the system mapping $\tilde{u}$ to $\tilde{y}$ is linear time invariant. We will denote this by $\tilde{\Sigma}$, the lifted system. What are the dimensions of the inputs and outputs. (In other words, by lifting the inputs and outputs, we got rid of the periodicity of the system and obtained a Multi-Input Multi-Output System).

(c) Give a state-space description of the lifted system. (Hint: Choose as a state variable $\tilde{x}(k) = x(kN)$, i.e., samples of the original state vector. Justify this choice).

(d) Show that the reachable subspace of the lifted system $\tilde{\Sigma}$ is included in the reachable subspace of the periodic system Σ. Show that the converse is true if the periodic system is reachable in T steps with $T = rN$ (a multiple of the period).

(e) Is it true that reachability of the periodic system Σ implies reachability of the lifted system $\tilde{\Sigma}$. Prove or show a counter example.