# Lectures on Dynamic Systems and Control

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# Chapter 28

# Stabilization: State Feedback

## 28.1 Introduction: Stabilization

One reason feedback control systems are designed is to stabilize systems that may be unstable. Although our earlier results show that a reachable but unstable system can have its state controlled by appropriate choice of control input, these results were obtained under some critical assumptions:

- the control must be unrestricted (as our reachability results assumed the control could be chosen freely);
- the system must be accurately described (i.e. we must have an accurate model of it);
- the initial state must be accurately known.

The trouble with unstable systems is that they are unforgiving when assumptions such as the above do not hold. Even if the first assumption above is assumed to hold, there will undoubtedly be modeling errors, such as improperly modeled dynamics or incompletely modeled disturbances (thus violating the second assumption). And even if we assume that the dynamics are accurately modeled, the initial state of the system is unlikely to be known precisely (violating the third assumption). It is thus clear that we need ongoing feedback of information about the state of the system, in order to have a hope of stabilizing an unstable system. Feedback can also improve the performance of a stable system (or, if the feedback is badly chosen, it can degrade the performance and possibly cause instability!). We shall come to understand these issues better over the remaining lectures.

How, then, can we design feedback controllers that stabilize a given system (or *plant* — the word used to describe the system that we are interested in controlling) ? To answer this, we have to address the issues of what kind of feedback variables are available for our controller. There are, in general, two types of feedback:

- state feedback
- output feedback.

With state feedback, all of the state variables (*e.g.,* $x$) of a system are available for use by the controller, whereas with output feedback, a set of output variables (*e.g.,* $y = Cx + Du$) related to the state variables are available. The state feedback problem is easier than the output feedback one, and richer in the sense that we can do more with control.

In the remainder of this chapter, we examine eigenvalue placement by state feedback. All our discussion here will be for the case of a *known* LTI plant. The issue of uncertainty and unmodeled dynamics should be dealt with as discussed in previous chapters; namely, by imposing a norm constraint on an appropriate closed loop transfer function. Our development in this lecture will use the notation of CT systems — but there is no essential difference for the DT case.

## 28.2 State Feedback

In the case of state feedback, we measure all of the state variables. Thus the plant specification is $(A, B, I, 0)$ — we omit the direct-feedthrough matrix, $D$, for simplicity, because including it would introduce only notational complications, without changing any conclusions. Our plant specification implies that the output equation is simply $y = x$. (In many applications, direct measurement of all system state variables is either impossible or impractical. We address the important topic of output feedback a little later in this lecture.)

For now, let us examine state feedback in further detail. Let our control, $u$, be specified by $u = Fx + v$, where $F$ is a constant matrix, and $v$ is an external input. This corresponds to LTI state feedback. Combining this control law with the state-space description for our $n$th-order plant, namely,

$$\delta x = Ax + Bu \qquad (28.1)$$

$$y = x \; , \qquad (28.2)$$

we find that the closed-loop dynamics are described by

$$\delta x = (A + BF)x + Bv \; , \qquad (28.3)$$

where we adopt the notation

$$\delta x = \begin{cases} \dot{x} & \text{for CT systems} \\ x(k+1) & \text{for DT systems} \end{cases} \; . \qquad (28.4)$$

As is apparent from (28.3), the closed-loop system is stable if and only if the eigenvalues of $A + BF$ are all in the stable region. In other words, $F$ stabilizes this system if and only if

$$\sigma(A + BF) \subset \begin{cases} \text{Open left half of the complex plane in continuous} - \text{time} \\ \text{open unit disc in discrete} - \text{time} \end{cases} \; , \qquad (28.5)$$

where $\sigma(A + BF)$ is the *spectrum* (set of eigenvalues) of $(A + BF)$.

A key question is: "Can $F$ be chosen so that the eigenvalues of $(A + BF)$ are placed at arbitrary desired locations?" The answer is provided by the following theorem.

**Theorem 28.1 (Eigenvalue Placement)** *There exists a matrix $F$ such that*

$$\det\left(\lambda I - [A + BF]\right) = \prod_{i=1}^{n} (\lambda - \mu_i) \qquad (28.6)$$

*for any arbitrary self-conjugate set of complex numbers $\mu_1, \ldots, \mu_n \in \mathbb{C}$ if and only if $(A, B)$ is reachable.*

**Proof.** To prove that reachability is necessary, suppose that $\lambda_i \in \sigma(A)$ is an unreachable mode. Let $w_i^T$ be the left eigenvector of $A$ associated with $\lambda_i$. It follows from the modal reachability test that $w_i^T A = \lambda_i w_i^T$ and $w_i^T B = 0$. Therefore,

$$\begin{aligned} w_i^T(A+BF) &= w_i^T A + (w_i^T B)F \\ &= w_i^T A + 0 \\ &= \lambda_i w_i^T \ . \end{aligned} \tag{28.7}$$

Equation (28.7) implies that $\lambda_i$ is an eigenvalue of $(A+BF)$ for any $F$. Thus, if $\lambda_i$ is an unreachable mode of the plant, then there exists no state feedback matrix $F$ that can move it.

We shall prove sufficiency for the single-input ($B = b$) case only. (The easiest proof for the multi-input case is, as in the single-input case below, based on a canonical form for reachable multi-input systems, which we have not examined in any detail, and this is why we omit the multi-input proof.) Since $(A, b)$ is reachable, there exists a similarity transformation $x = Tz$ such that $T^{-1}AT$ and $T^{-1}b$ have the controller canonical form

$$\tilde{A} = T^{-1}AT = \begin{bmatrix} -\alpha_1 & -\alpha_2 & \cdots & -\alpha_n \\ 1 & 0 & \cdots & 0 \\ & & \ddots & \\ & & 1 & 0 \end{bmatrix} \tag{28.8}$$

$$\tilde{b} = T^{-1}b = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \ . \tag{28.9}$$

Recall that the coefficients $\alpha_i$ in the matrix $\tilde{A}$ define the characteristic polynomial of $\tilde{A}$ and $A$:

$$\alpha(\lambda) = \lambda^n + \alpha_1 \lambda^{n-1} + \cdots + \alpha_n \tag{28.10}$$

Let

$$\prod_{i=1}^{n} (\lambda - \mu_i) = \lambda^n + \alpha_1^d \lambda^{n-1} + \cdots + \alpha_n^d = \alpha^d(\lambda) \ . \tag{28.11}$$

If $u = \tilde{F}z$ with $\tilde{F}$ being the row vector

$$\tilde{F} = \begin{bmatrix} \tilde{f}_1 & \cdots & \tilde{f}_n \end{bmatrix}$$

then

$$\tilde{A} + \tilde{b}\tilde{F} = \begin{bmatrix} -\alpha_1 + \tilde{f}_1 & -\alpha_2 + \tilde{f}_2 & \cdots & -\alpha_n + \tilde{f}_n \\ 1 & 0 & \cdots & 0 \\ & \ddots & & \\ & & 1 & 0 \end{bmatrix} \ . \tag{28.12}$$

It is evident that we simply have to choose $\tilde{f}_i = -\alpha_i^d + \alpha_i$ for $i = 1, \ldots, n$ to get the desired closed-loop characteristic polynomial $\alpha^d(\lambda)$.

We have thus been able to place the eigenvalues in the desired locations. Now, using the similarity transformation and $\tilde{F}$, we must find $F$ so that $A + bF$ has the same eigenvalues. Since $u = \tilde{F}z$ and

$x = Tz$, $u = \tilde{F}T^{-1}x$. Thus we should define $F = \tilde{F}T^{-1}$. (Verify that $A + bF$ has the same eigenvalues as $\tilde{A} + \tilde{b}\tilde{F}$.) This completes the proof.

The calculation that was described above of a feedback matrix that places the poles of $A + bF$ at the roots a specified polynomial $\alpha^d(s)$ can be succinctly represented in a simple formula. The matrix $A$ and $\tilde{A}$ have the same characteristic polynomial, $\alpha(\lambda)$, which implies that $\tilde{A}$ satisfies

$$(\tilde{A})^n = -\alpha_1\tilde{A}^{n-1} - \alpha_2\tilde{A}^{n-2} - \ldots - \alpha_n I.$$

Based on the above relation the desired characteristic polynomial satisfies

$$\begin{aligned}
\alpha^d(\tilde{A}) &= \tilde{A}^n + \alpha_1^d\tilde{A}^{n-1} + \alpha_2^d\tilde{A}^{n-2} + \ldots + \alpha_n^d I, \\
&= (\alpha_1^d - \alpha_1)\tilde{A}^{n-1} + (\alpha_2^d - \alpha_2)\tilde{A}^{n-2} + \ldots + (\alpha_n^d - \alpha_n)I.
\end{aligned}$$

We define the unit vectors $e_i^T$, $i = 1, 2, \ldots, n$ as

$$e_i^T = \begin{bmatrix} 0 & 0 & \ldots & 0 & \overset{i^{th} position}{1} & 0 & \ldots & 0 \end{bmatrix}.$$

Due to the special structure of the matrix $\tilde{A}$ the reader should be able to check that

$$\begin{aligned}
e_n^T\alpha^d(\tilde{A}) &= (\alpha_1^d - \alpha_1)e_n^T\tilde{A}^{n-1} + (\alpha_2^d - \alpha_2)e_n^T\tilde{A}^{n-2} + \ldots + (\alpha_n^d - \alpha_n)e_n^T I \\
&= (\alpha_1^d - \alpha_1)e_1^T + (\alpha_2^d - \alpha_2)e_2^T + \ldots + (\alpha_n^d - \alpha_n)e_n^T \\
&= -\tilde{F}.
\end{aligned}$$

Recall that the transformation $T$ that transforms a system into reachable form is given by $T = R_n\widetilde{R_n}^{-1}$ where

$$\begin{aligned}
R_n &= \begin{bmatrix} b & Ab & \ldots & A^{n-1}b \end{bmatrix}, \\
\widetilde{R_n} &= \begin{bmatrix} \tilde{b} & \tilde{A}\tilde{b} & \ldots & \tilde{A}^{n-1}\tilde{b} \end{bmatrix}.
\end{aligned}$$

The matrix $\widetilde{R_n}$ has the following form

$$\widetilde{R_n} = \begin{bmatrix} 1 & * & * & \ldots \\ 0 & 1 & * & \ldots \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{bmatrix}, \qquad (28.13)$$

where $*$ denotes entries that can be different from zero. The feedback matrix $F$ is related to $\tilde{F}$ via the relation $F = \tilde{F}T^{-1}$ which implies that

$$\begin{aligned}
F &= \tilde{F}T^{-1} \\
&= -e_n^T\alpha^d(\tilde{A})T^{-1} \\
&= -e_n^T\alpha^d(T^{-1}AT)T^{-1} \\
&= -e_n^T T^{-1}\alpha^d(A)TT^{-1} \\
&= -e_n^T\widetilde{R_n}{R_n}^{-1}\alpha^d(A).
\end{aligned}$$

Note that from Equation 28.13 we have $e_n^T\widetilde{R_n} = e_n^T$, which results in the following formula, which is commonly called Ackermann's formula

$$F = -e_n^T{R_n}^{-1}\alpha^d(A). \qquad (28.14)$$

Some comments are in order:

1. If $(A, B)$ is not reachable, then the reachable modes, and only these, can be changed by state feedback.

2. The pair $(A, B)$ is said to be *stabilizable* if its unreachable modes are all stable, because in this case, and only in this case, $F$ can be selected to change the location of all unstable modes to stable locations.

3. Despite what the theorem says we can do, there are good practical reasons why one might temper the application of the theorem. Trying to make the closed-loop dynamics very fast generally requires large $F$, and hence large control effort — but in practice there are limits to how much control can be exercised. Furthermore, unmodeled dynamics could lead to instability if we got too ambitious with our feedback.

   The so-called *linear-quadratic regulator* or LQR formulation of the controller problem for linear systems uses an integral-square (i.e. quadratic) cost criterion to pose a compromise between the desire to bring the state to zero and the desire to limit control effort. In the LTI case, and with the integral extending over an infinite time interval, the optimal control turns out to be precisely an LTI state feedback. The solution of the LQR problem for this case enables computation of the optimal feedback gain matrix $F^*$ (most commonly through the solution of an algebraic Riccati equation). You are led through some exploration of this on the homework. See also the article on "Linear Quadratic Regulator Control" by Lublin and Athans in *The Control Handbook*, W.S. Levine (Ed.), CRC Press, 1996.

4. State feedback cannot change reachability, but it can affect observability — either destroying it or creating it.

5. State feedback can change the poles of an LTI system, but does not affect the zeros (unless the feedback happens to induce unobservability, in which case what has occurred is that a pole has been shifted to exactly cancel a zero). Note that, if the open-loop and closed-loop descriptions are minimal, then their transmission zeros are precisely the values of $s$ where their respective system matrices drop rank. These system matrices are related by a nonsingular transformation:

$$\begin{pmatrix} sI - (A + BF) & -B \\ C & 0 \end{pmatrix} = \begin{pmatrix} sI - A & -B \\ C & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ F & I \end{pmatrix} \tag{28.15}$$

   Hence the closed-loop and open-loop zeros are identical. (We omit a more detailed discussion of what happens in the nonminimal case.)

**Example 28.1 Inverted Pendulum**

A cart of mass $M$ slides on a frictionless surface. The cart is pulled by a force $u(t)$. On the cart a pendulum of mass $m$ is attached via a frictionless hinge, as shown in Figure 28.1. The pendulum's center of mass is located at a distance $l$ from either end. The moment of inertia of the pendulum about its center of mass is denoted by $I$. The position of the center of mass of the cart is at a distance $s(t)$ from a reference point. The angle $\theta(t)$ is the angle that the pendulum makes with respect to the vertical axis which is assumed to increase clockwise.

First let us write the equations of motion that result from the free-body diagram of the cart. The vertical forces $P$, $R$ and $Mg$ balance out. For the horizontal forces we have the following equation

$$M\ddot{s} = u - N. \tag{28.16}$$

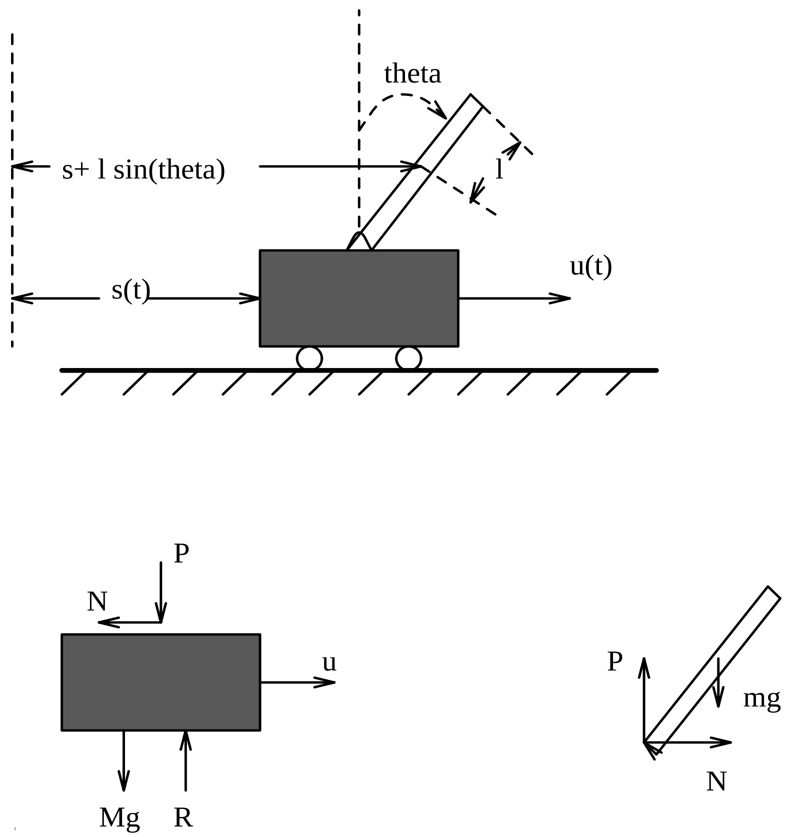


Figure 28.1: Inverted Pendulum

From the free-body diagram of the pendulum, the balance of forces in the horizontal direction gives the equation

$$\begin{aligned} m\frac{d^2}{dt^2}\left(s + l\sin(\theta)\right) &= N,, \\ m\frac{d}{dt}\left(\dot{s} + l\cos(\theta)\dot{\theta}\right) &= N, \\ m\left(\ddot{s} - l\sin(\theta)(\dot{\theta})^2 + l\cos(\theta)\ddot{\theta}\right) &= N, \end{aligned} \tag{28.17}$$

and the balance of forces in the vertical direction gives the equation

$$\begin{aligned} m\frac{d^2}{dt^2}\left(l\cos(\theta)\right) &= P - mg, \\ m\frac{d}{dt}\left(-l\sin(\theta)\dot{\theta}\right) &= P - mg, \\ m\left(-l\cos(\theta)(\dot{\theta})^2 - l\sin(\theta)\ddot{\theta}\right) &= P - mg. \end{aligned} \tag{28.18}$$

Finally by balancing the moments around the center of mass we get the equation

$$I\ddot{\theta} = Pl\sin(\theta) - Nl\cos(\theta). \tag{28.19}$$

From equations 28.16, 28.17 we can eliminate the force $N$ to obtain

$$(M+m)\ddot{s} + m\left(l\cos(\theta)\ddot{\theta} - l\sin(\theta)(\dot{\theta})^2\right) = u. \tag{28.20}$$

Substituting equations 28.17, 28.18 into equation 28.19 gives us

$$\begin{aligned} I\ddot{\theta} &= l\left(mg - ml\cos(\theta)(\dot{\theta})^2 - ml\sin(\theta)\ddot{\theta}\right)\sin(\theta) \\ &- l\left(m\ddot{s} - ml\sin(\theta)(\dot{\theta})^2 + ml\cos(\theta)\ddot{\theta}\right)\cos(\theta). \end{aligned}$$

Simplifying the above expression gives us the equation

$$(I+ml^2)\ddot{\theta} = mgl\sin(\theta) - ml\ddot{s}\cos(\theta). \tag{28.21}$$

The equations that describe the system are 28.20 and 28.21. We can have a further simplification of the system of equations by removing the term $\ddot{\theta}$ from equation 28.20, and the term $\ddot{s}$ from equation 28.21. Define the constants

$$\begin{aligned} M_t &= M+m \\ L &= \frac{I+ml^2}{ml}. \end{aligned}$$

Substituting $\ddot{\theta}$ from equation 28.21 into equation 28.20 we get

$$\left(1 - \frac{ml}{M_tL}\cos(\theta)^2\right)\ddot{s} + \frac{ml}{M_tL}g\sin(\theta)\cos(\theta) - \frac{ml}{M_t}\sin(\theta)(\dot{\theta})^2 = \frac{1}{M_t}u. \tag{28.22}$$

Similarly we can substitute $\ddot{s}$ from equation 28.20 into equation 28.21 to get

$$\left(1 - \frac{ml}{M_tL}\cos(\theta)^2\right)\ddot{\theta} - \frac{g}{L}\sin(\theta) + \frac{ml}{M_tL}\sin(\theta)\cos(\theta)(\dot{\theta})^2 = -\frac{1}{M_tL}\cos(\theta)u. \tag{28.23}$$

These are nonlinear equations due to the presence of the terms $\sin(\theta)$, $\cos(\theta)$, and $(\dot{\theta})^2$. We can linearize these equations around $\theta = 0$ and $\dot{\theta} = 0$, by assuming that $\theta(t)$ and $\dot{\theta}(t)$ remain small. Recall that for small $\theta$

$$\begin{aligned} \sin(\theta) &\approx \theta - \frac{1}{6}\theta^3 \\ \cos(\theta) &\approx 1 - \frac{1}{2}\theta^2, \end{aligned}$$

and using these relations we can linearize the equations 28.22 and 28.23. The linearized system of equations take the form

$$\begin{aligned} \left(1 - \frac{ml}{M_tL}\right)\ddot{s} + \frac{ml}{M_tL}\frac{g}{L}\theta &= \frac{1}{M_t}u, \\ \left(1 - \frac{ml}{M_tL}\right)\ddot{\theta} - \frac{g}{L}\theta &= -\frac{1}{M_tL}u. \end{aligned}$$

Choose the following state variables

$$x = \begin{bmatrix} s \\ \dot{s} \\ \theta \\ \dot{\theta} \end{bmatrix},$$

to write a state space model for the invert ed pendulum. Using these state variables the following state space model can be easily obtained

$$\begin{aligned}
\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -\alpha\frac{ml}{M_t L}g & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \alpha\frac{g}{L} & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{\alpha}{M_t} \\ 0 \\ -\frac{\alpha}{LM_t} \end{pmatrix} u \\
y &= \begin{bmatrix} 1 & 0 & & 0 \end{bmatrix} x,
\end{aligned}$$

where the constant $\alpha$ is given by

$$\alpha = \frac{1}{\left(1 - \frac{ml}{M_t L}\right)}.$$

Intuitively it is clear that the equilibrim point $[s = \text{constant}, \dot{s} = 0, \theta = 0, \dot{\theta} = 0]$ is an unstable equilibrim point. To verify this we compute the eigenvalues of the matrix $A$ by solving the equation $\det(\lambda I - A) = 0$. The eigenvalues are

$$\left\{ \begin{matrix} 0 & 0 & \sqrt{\frac{\alpha g}{L}} & -\sqrt{\frac{\alpha g}{L}} \end{matrix} \right\}.$$

Therefore we have two eigenvalues at the $j\omega$ axis and one eigenvalue in the open right half of the complex plane, which indicates instability.

Now let us consider the case where $M = 2kg$, $m = .1kg$, $l = .5m$, $I = .025kgm^2$, and of course $g = 9.8m/s^2$. Assume that we can directly measure the state variables, $s$, $\dot{s}$, $\theta$ and $\dot{\theta}$. We want to design a feedback control law $u = F\hat{x} + r$ to stabilize this system. In order to do that we will choose a feeback matrix $F$ to place the poles of the closed-loop system at $\{-1, -1, -3, -3\}$. Using Ackermann's formula

$$F = -\begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} {R_n}^{-1} \alpha^d(A)$$

where $\alpha^d(\lambda) = (\lambda + 1)(\lambda + 1)(\lambda + 3)(\lambda + 3)$ which is the ploynomial whose roots are the desired new pole locations, and $R_n$ is the *reachability matrix.* In specific using the parameters of the problem we have

$$\begin{aligned}
F &= -\begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0.4878 & 0 & 0.1166 \\ 0.4878 & 0 & 0.1166 & 0 \\ 0 & -0.4878 & 0 & -4.8971 \\ -0.4878 & 0 & -4.8971 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 9.0 & 24.0 & -7.7 & -1.9 \\ 0 & 9.0 & -24.9 & -7.7 \\ 0 & 0 & 330.6 & 104.3 \\ 0 & 0 & 1047.2 & 330.6 \end{bmatrix} \\
&= \begin{bmatrix} 1.8827 & 5.0204 & 67.5627 & 21.4204 \end{bmatrix}
\end{aligned}$$

The closed-loop system is given by

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 & 1.0 & 0 & 0 \\ 0.9184 & 2.449 & 32.7184 & 10.449 \\ 0 & 0 & 0 & 1.0 \\ -0.9184 & -2.4490 & -22.9184 & -10.4490 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0.4878 \\ 0 \\ -0.4878 \end{bmatrix} r$$

In Figure 28.2 we show the time trajectories of the closed-loop linearized system when the reference input $r(t)$ is identically zero and the intial angular displacement of the pendulum

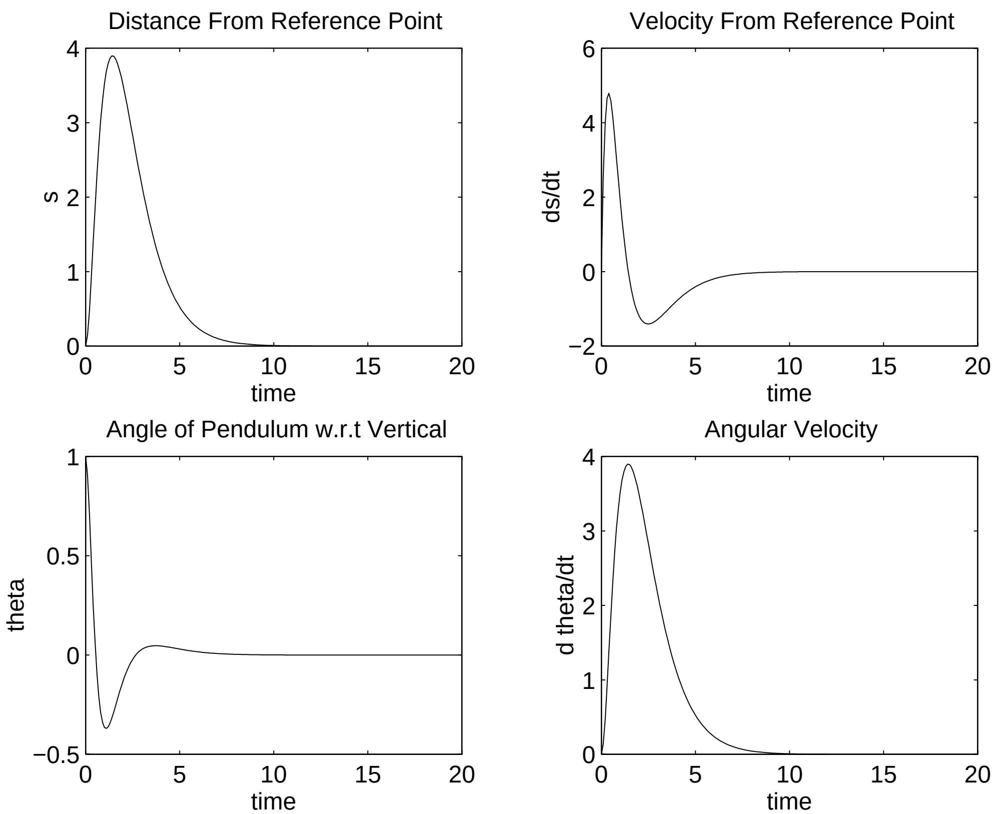


Figure 28.2: Plot of the State Variables of the Closed-Loop Linearized System with $r = 0$ and the Initial Condition $s = 0$, $\dot{s} = 0$, $\theta = 1.0$, and $\dot{\theta} = 0$

is 1.0 radians. In this simulation the initial conditions on all the other state variables are zero.

We can also look at the performance of this controller if it is applied to the nonlinear system. In this case we should simulate the dynamics of the following nonlinear system of equations

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{mlg}{M_t L}\frac{1}{\alpha(x_3)}\sin(x_3)\cos(x_3) + \frac{ml}{M_t}\frac{1}{\alpha(x_3)}\sin(x_3)(x_4)^2 \\ x_4 \\ \frac{g}{L}\frac{1}{\alpha(x_3)}\sin(x_3) - \frac{ml}{M_t L}\frac{1}{\alpha(x_3)}\sin(x_3)\cos(x_3)(x_4)^2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M_t}\frac{1}{\alpha(x_3)} \\ 0 \\ -\frac{1}{M_t L}\frac{\cos(x_3)}{\alpha(x_3)} \end{bmatrix} u$$

$$u = \begin{bmatrix} 1.8827 & 5.0204 & 67.5627 & 21.4204 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + r,$$

where $\alpha(x_3)$ is defined as

$$\alpha(x_3) = \left(1 - \frac{ml}{M_t L}\cos(x_3)^2\right).$$

In Figure 28.3 we show the time trajectories of the nonlinear closed-loop system when the reference input $r(t)$ is identically zero and the intial angular displacement of the pendulum is 1.0 radians. In this simulation the initial conditions on all the other state variables are zero.

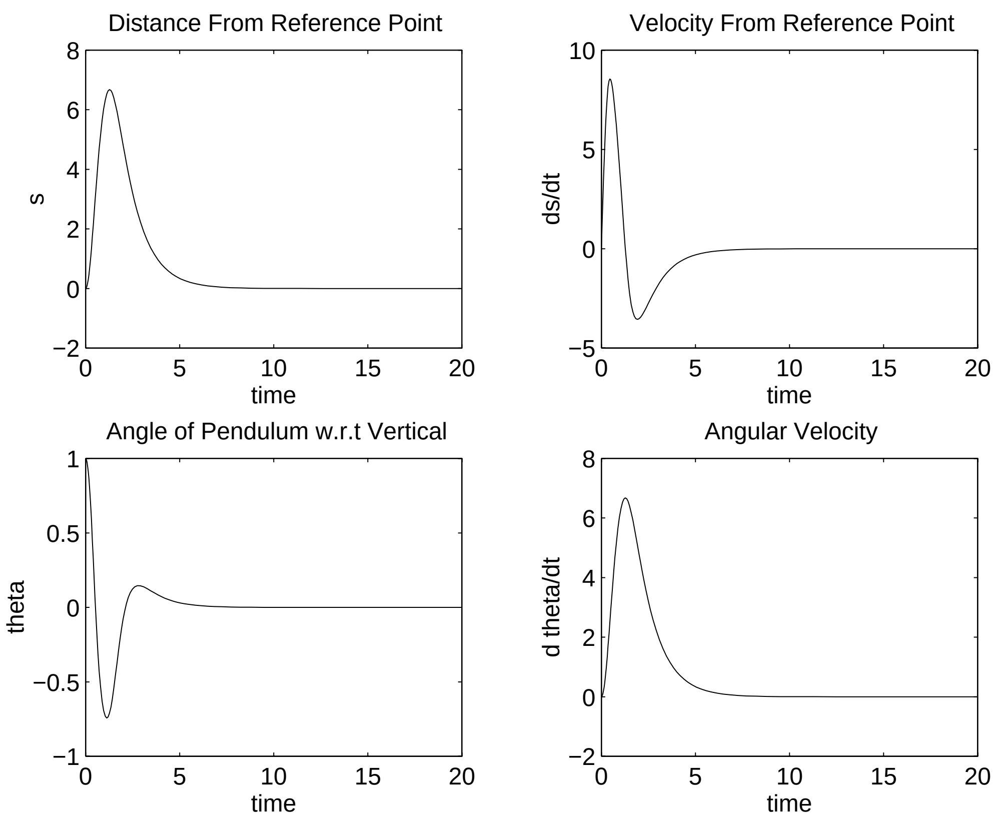


Figure 28.3: Plot of the State Variables of the Nonlinear Closed-Loop System with $r = 0$ and the Initial Condition $s = 0$, $\dot{s} = 0$, $\theta = 1.0$, and $\dot{\theta} = 0$

# Exercises

**Exercise 28.1** Let $(A, B, C, 0)$ be a reachable and observable LTI state-space description of a discrete-time or continuous-time system. Let its input $u$ be related to its output $y$ by the following *output feedback* law:

$$u = Fy + r$$

for some constant matrix $F$, where $r$ is a new external input to the closed-loop system that results from the output feedback.

**(a)** Write down the state-space description of the system mapping $r$ to $y$.

**(b)** Is the new system reachable? Prove reachability, or show a counterexample.

**(c)** Is the new system observable? Prove observability, or show a counterexample.

**Exercise 28.2 (Discrete Time "Linear-Quadratic" or LQ Control)**
Given the *linear* system $x_{i+1} = Ax_i + Bu_i$ and a specified initial condition $x_0$, we wish to find the sequence of controls $u_0, u_1, \ldots, u_N$ that minimizes the *quadratic* criterion

$$J_{0,N}(x_0, u_0, \ldots, u_N) = \sum_{0}^{N} (x_{i+1}^T Q x_{i+1} + u_i^T R u_i)$$

Here $Q$ is positive semi-definite (and hence of the form $Q = V^T V$ for some matrix $V$) and $R$ is positive definite (and hence of the form $R = W^T W$ for some *nonsingular* matrix $W$). The rationale for this criterion is that it permits us, through proper choice of $Q$ and $R$, to trade off our desire for small state excursions against our desire to use low control effort (with state excursions and control effort measured in a sum-of-squares sense). This problem will demonstrate that the optimal control sequence for this criterion has the form of a time-varying *linear state feedback*.

Let the optimal control sequence be denoted by $u_0^*, \ldots, u_N^*$, let the resulting state sequence be denoted by $x_1^*, \ldots, x_{N+1}^*$, and let the resulting value of $J_{0,N}(x_0, u_0, \ldots, u_N)$ be denoted by $L_{0,N}^*(x_0)$.

**(a)** Argue that $u_k^*, \ldots, u_N^*$ is also the the sequence of controls $u_k, \ldots, u_N$ that minimizes $J_{k,N}(x_k^*, u_k, \ldots, u_N)$, $0 \le k \le N$. [This observation, in its general form, is termed the *principle of optimality*, and underlies the powerful optimization framework of *dynamic programming*.]

**(b)** Show that

$$J_{k,N}(x_k, u_k, \ldots, u_N) = \sum_{k}^{N} \|e_\ell\|^2$$

where $e_\ell = Cx_\ell + Du_\ell$ and

$$C = \begin{pmatrix} VA \\ 0 \end{pmatrix}, \qquad D = \begin{pmatrix} VB \\ W \end{pmatrix}$$

**(c)** Let

$$\mathcal{U}_{k,N} = \begin{pmatrix} u_k \\ \vdots \\ u_N \end{pmatrix}, \qquad \mathcal{E}_{k,N} = \begin{pmatrix} e_k \\ \vdots \\ e_N \end{pmatrix}$$

Show that $\mathcal{E}_{k,N} = \mathcal{C}_{k,N} x_k + \mathcal{D}_{k_N} \mathcal{U}_{k,N}$ for appropriate matrices $\mathcal{C}_{k,N}$ and $\mathcal{D}_{k,N}$, and show that $\mathcal{D}_{k,N}$ has full column rank.

**(d)** Note from (b) that $J_{k,N}(x_k, u_k, \ldots, u_N) = \|\mathcal{E}_{k,N}\|^2$. Use this and the results of (a), (c) to show that

$$\mathcal{U}_{k,N}^* = -(\mathcal{D}_{k,N}^T \mathcal{D}_{k,N})^{-1} \mathcal{D}_{k,N}^T \mathcal{C}_{k,N} x_k^*$$

and hence that $u_k^* = F_k^* x_k^*$ for some state feedback gain matrix $F_k^*$, $0 \leq k \leq N$. The optimal control sequence thus has the form of a time-varying linear state feedback.

**(e)** Assuming the optimal control problem has a solution for $N = \infty$, argue that in this "infinite-horizon" case the optimal control is given by $u_k^* = F^* x_k^*$ for a *constant* state feedback gain matrix $F^*$.

**Exercise 28.3 (Continuous-Time LQ Control)**

Consider the controllable and observable system $\dot{x}(t) = Ax(t) + Bu(t)$, $y(t) = Cx(t)$. It can be shown that the control which minimizes

$$J = \int_0^\infty [\, y'(t)y(t) + u'(t)Ru(t) \,]\, dt$$

with $R$ positive definite, is of the form $u(t) = F^* x(t)$, where

$$F^* = -R^{-1} B' P \tag{2.1}$$

and where $P$ is the unique, symmetric, positive definite solution of the following equation (called the *algebraic Riccati equation* or ARE):

$$PA + A'P + Q - PBR^{-1}B'P = 0\ , \quad Q = C'C \tag{2.2}$$

The control is guaranteed to be stabilizing. The significance of $P$ is that the minimum value of $J$ is given by $x'(0)Px(0)$.

In the case where $u$ and $y$ are scalar, so $R$ is also a scalar which we denote by $r$, the optimum closed-loop eigenvalues, i.e. the eigenvalues of $A + BF^*$, can be shown to be the left-half-plane roots of the so-called *root square characteristic polynomial*

$$a(s)a(-s) + p(s)r^{-1}p(-s)$$

where $a(s) = \det(sI - A)$ and $p(s)/a(s)$ is the transfer function from $u$ to $y$.

Now consider the particular case where

$$A = \begin{pmatrix} 0 & 1 \\ 9 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

This could represent a magnetic suspension scheme with actuating current $u$ and position $y$ (or a simplified model of an inverted pendulum).

**(a)** Show that the system is unstable.

**(b)** Find the transfer function from $u$ to $y$.

**(c)** Using the root square characteristic polynomial for this problem, approximately determine in terms of $r$ the optimum closed-loop eigenvalues, assuming $r \ll 1$.

**(d)** Determine the optimum closed-loop eigenvalues for $r \to \infty$, and find the $F^*$ that gives this set of eigenvalues.

**(e)** Verify the result in (d) by computing the optimal gain $F^*$ via the formulas in (2.1) and (2.2). (In order to get a meaningful solution of the ARE, you should *not* set $r^{-1} = 0$, but still use the fact that $r \gg 1$.)

**Exercise 28.4 (Eigenstructure Assignment)** Let $(A, B, I, 0)$ be an $m$-input, reachable, $n$th-order LTI system. Let the input be given by the LTI state feedback

$$u = Fx$$

Suppose we desire the new, closed-loop eigenvalues to be $\mu_i$, with associated eigenvectors $p_i$. We have seen that the $\mu_i$ can be placed arbitrarily by choice of $F$ (subject only to the requirement that they be at self-conjugate locations, i.e. for each complex value we also select its conjugate). Assume in this problem that none of the $\mu_i$'s are eigenvalues of $A$.

**(a)** Show that the eigenvector $p_i$ associated with $\mu_i$ must lie in the $m$-dimensional subspace $\mathcal{R}a[(\mu_i I - A)^{-1}B]$, i.e.,

$$p_i = (\mu_i I - A)^{-1} B q_i$$

for some $q_i$ .

**(b)** Show that if $p_1, \ldots, p_n$ are a set of attainable, linearly independent, closed-loop eigenvectors, then

$$F = [q_1, \ \ldots, \ q_n] \ [p_1, \ \ldots, \ p_n]^{-1}$$

where $q_i, \ldots, q_n$ are as defined in (a).

**(c)** Verify that specifying the closed-loop eigenvalues and eigenvectors, subject to the restrictions in (a), involves specifying exactly $nm$ numbers, which matches the number of free parameters in $F$.