MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science

6.241: Dynamic Systems—Fall 2002

Recitation 11

Reachability and Observability

In this recitation, we review the definitions and the tests for reachability and observability for finite dimensional LTI systems. We then present several examples.

Definitions

Consider the n-dimensional LTI system, $x_+ = Ax + Bu$, $y = Cx + Du$.

Reachability	Observability
A state d is reachable in time L (CT) (or k steps for DT) if there is an input $u(t) \in [0, L]$ (or $[0, k-1]$) that transfers the state x(t) from the origin to d in time L (or k), i.e., $x(L) = d$ (or $x(k) = d$).	A state q is *un*observable over $[0, T)$ (or k steps) if when $x(0) = q$ and $\forall u(t)$ on $[0, T)$ (or $[0, k-1]$), the output $y(t)$ for $x(0) = q$ is the same as the output for $x(0) = 0$.
The reachable in time L (or k) subspace is the set of all states d reachable in time L (or k).	The unobservable in time T (or k) subspace is the set of all states q that are unobservable in time T (or k).
A system is reachable in time L (or k) if every state d in the state space is reachable in time L (or k).	A system is observable in time T (or k) if the only state that is unobservable in time T (or k) is the zero state, $q = 0$.

Reachability and Observability Tests

The following tests are equivalent tests for reachability (or observability).

	Reachability	Observability
	A system is reachable iff	A system is observable iff
1.	Rank$\begin{bmatrix} A^{n-1}B & A^{n-2}B & \ldots & B \end{bmatrix} = n.$	Rank$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n.$
2a.	There does not exist $w \neq 0$ such that $w^T A = \lambda w^T$ and $w^T B = 0$.	There does not exist $v \neq 0$ such that $Av = \lambda v$ and $Cv = 0$.
2b.	Rank$\begin{bmatrix} sI - A & B \end{bmatrix} = n, \forall s \in \mathbb{C}$	Rank$\begin{bmatrix} sI - A \\ C \end{bmatrix} = n, \forall s \in \mathbb{C}.$
	Here, we only need to check for $s_i = \lambda_i(A)$	
3.	DT rank$(P_n) = n$, $P_k = \sum_{i=0}^{k-1} A^i BB^T (A^i)^T$ CT rank$(P_L) = n$, $P_t = \int_0^t e^{A\tau} BB^T (e^{A\tau})^T \mathrm{d}\tau$	DT rank$(Q_n) = n$, $Q_k = \sum_{i=0}^{k-1} (A^i)^T C^T C A^i$ CT rank$(Q_L) = n$, $Q_t = \int_0^t (e^{A\tau})^T C^T C e^{A\tau} \mathrm{d}\tau$

Examples

Example 1: This is a simple example to verify that the three reachability tests listed above indeed give identical results. Suppose $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

(a) Rank$\begin{bmatrix} B & AB \end{bmatrix}$ =Rank$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = 1 < 2$.

(b) Rank$\begin{bmatrix} sI - A & B \end{bmatrix}$ =Rank$\begin{bmatrix} s & -1 & 1 \\ 0 & s & 0 \end{bmatrix} = 1$ for $s = 0$. (Note that $s = 0$ is an eigenvalue of A.)

(c) $P_L = \int_0^L e^{A\tau} BB^T (e^{A\tau})^T \mathrm{d}\tau$, where $e^{A\tau} = I + A\tau = \begin{bmatrix} 1 & \tau \\ 0 & 1 \end{bmatrix}$ and $e^{A\tau} B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. So, $P_L = \int_0^L \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \mathrm{d}\tau = \begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix}$. Thus, rank$(P_L) = 1 < 2$.

Example 2: Suppose $A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & a & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ 1 \\ b \end{bmatrix}$. For which values of a and b is the system unreachable? We have that $R_3 = \begin{bmatrix} B & AB & A^2 B \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1+a \\ 1 & a & 2+a^2 \\ b & 0 & 0 \end{bmatrix}$. This matrix looses rank when $b = 0$ (and a is arbitrary) or when $a = 2$ and b is arbitrary.

Example 3: (From [[1] page 238]). Consider the scalar system $\dot{x} = -ax + bu$. Determine a $u(t)$ that will transfer the state from $x(0) = x_o$ to the origin in L seconds, i.e., $x(L) = 0$.

In general, recall that $x(L) = e^{AL} x(0) + \int_0^L e^{(L-\tau)A} Bu(\tau) \mathrm{d}\tau$. Suppose $d - e^{AL} s \in Ra(P_L)$. That is, $d - e^{AL}s = P_L\alpha$ for some α. If we pick $u(t) = B^T(e^{(L-t)A})^T\alpha$, we have that $x(L) = e^{AL}x(0) + \int_0^L e^{(L-\tau)A}BB^T(e^{(L-t)A})^T\alpha \mathrm{d}\tau = e^{AL}x(0) + \int_0^L e^{(L-\tau)A}BB^T(e^{(L-t)A})^T \mathrm{d}\tau\alpha = e^{AL}x(0) + P_L\alpha = e^{AL}x(0) + d - e^{AL}s$. So, if $x(0) = s$, due to our choice of $u(t)$ we have $x(L) = d$.

Now, if $\text{rank}(P_L) = n$, then $\alpha = P_L^{-1}(d - e^{AL}s)$, so the input $u(t) = B^T(e^{(L-t)A})^T P_L^{-1}(d - e^{AL}s)$ will transfer the initial state $x(0) = s$ to the state d in L seconds, i.e., $x(L) = d$.

Back to our example, we first note that the system is reachable, and that

$$P_L = \int_0^L e^{-(L-\tau)a}bbe^{-(L-\tau)a}\mathrm{d}\tau = \frac{b^2}{2a}[1 - e^{-2aL}].$$

Using the formula above, $u(t) = -\frac{2a}{b}\frac{1}{e^{2aL}-1}e^{at}x_o$. Finally, we check that with the chosen $u(t)$, $x(t) = e^{At} + \int_0^t e^{A(t-\tau)}Bu(\tau)\mathrm{d}\tau = e^{-at}\left[1 - \frac{e^{2at}-1}{e^{2aL}-1}\right]x_o$, and $x(L) = 0$ as desired.

Example 4: (From [[1] page 266]). Given $A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$. Put the system $\dot{x} = Ax + Bu$ in the standard form for unreachable systems. That is, we want to find the transformation $T = \begin{bmatrix} T_1 & T_2 \end{bmatrix}$ such that $\bar{A} = T^{-1}AT = \begin{bmatrix} A_1 & A_{12} \\ 0 & A_2 \end{bmatrix}$ and $\bar{B} = T^{-1}B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}$. Recall that the columns of T_1 form a basis for the reachable subspace, and recall that the range of R_n is the reachable subspace, so, we want $Ra(T_1) = Ra(R_n)$. For our matrices, $R_3 = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & -1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & -1 & 0 & 1 \end{bmatrix}$. So, the reachable subspace is spanned by the vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$, so let $T_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$. Now, we complete T by choosing a T_2 that has columns that are linearly independent of the columns of T_1. In this case, one choice is $T_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Using this T, we have that $\bar{A} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$ and $\bar{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$. Verify that A_1 and B_1 are reachable.

References

[1] Antsakalis, P.J., Michel A.N., "Linear Systems." *McGraw Hill*: 1998.