MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science

6.241: Dynamic Systems—Fall 2003

HOMEWORK 7 SOLUTIONS

Exercise 14.1 a) If Δ is real, then $A + \Delta \in \mathbf{R}$. When eigenvalues of $(A + \Delta)$ are at $j\omega_0$, then $(A + \Delta) - j\omega_0 I$ is singular. That implies that

$$\exists \underline{v} \neq 0 \text{ in } \mathbf{C} \text{ s.t.} ((A + \Delta) - j\omega_0 I)\underline{v} = 0.$$

This further implies that

$$\begin{aligned}
((A+\Delta) - j\omega_0 I)(\underline{v}_R + j\underline{v}_I) &= \underline{0} \\
\rightarrow ((A+\Delta)\underline{v}_R + (\omega_0 I \underline{v}_I)) + j((A+\Delta)\underline{v}_I - \omega_0 I \underline{v}_R) &= \underline{0} + \underline{0} * j \\
\rightarrow \begin{pmatrix} A+\Delta & \omega_0 I \\ -\omega_0 I & A+\Delta \end{pmatrix} \begin{pmatrix} \underline{v}_R \\ \underline{v}_I \end{pmatrix} &= \begin{pmatrix} \underline{0} \\ \underline{0} \end{pmatrix} \\
\rightarrow \left(\begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix} + \begin{pmatrix} A & \omega_0 I \\ -\omega_0 I & A \end{pmatrix} \begin{pmatrix} \underline{v}_R \\ \underline{v}_I \end{pmatrix} \right) &= \begin{pmatrix} \underline{0} \\ \underline{0} \end{pmatrix} \\
(\tilde{\Delta} + A_\omega)\underline{x} &= \underline{0}.
\end{aligned}$$

However, note that $\|\tilde{\Delta}\| = \|\Delta\|$. Thus, the smallest $\tilde{\Delta}$ that makes $\tilde{\Delta} + A_\omega$ singular is

$$\min \|\tilde{\Delta}\| = \min \|\Delta\| = \gamma(A) \geq \sigma_{min}(A_\omega).$$

It can be seen as an additive perturbation.

b) Note that when $\tilde{\Delta}$ has the structure above, it will cause A_ω to lose rank twice (i.e. its rank drops from $2n$ to $2n-2$), and this corresponds to the fact that both $det(j\omega_0 - (A + \Delta)) = 0$ and $det(-j\omega_0 - (A + \Delta) = 0$ simultaneously (both complex conjugate eigenvalues hit the $j\omega$-axis). Now recall from problem 5.2, that the smallest size $\tilde{\Delta}$ that will cause A_ω to lose rank by 2 must have a size $\sigma_{(2n-2)+1}(A_\omega)$, i.e. $\min\{\|\tilde{\Delta}\|_2 \mid rank(A_\omega + \tilde{\Delta}) \leq r\} = \sigma_{r+1}(A_\omega)$, and this is true for a delta with no structure imposed on it. So, when $\tilde{\Delta}$ does have structure, we have that $\min\{\|\tilde{\Delta}\|_2 \mid rank(A_\omega + \tilde{\Delta}) \leq 2n-2\} \geq \sigma_{2n-1}(A_\omega)$.

Alternatively, Consider $A'_\omega A_\omega$.

$$\begin{pmatrix} A' & -\omega_0 I \\ \omega_0 & A' \end{pmatrix} \begin{pmatrix} A & \omega_0 I \\ -\omega_0 I & A \end{pmatrix} = \begin{pmatrix} A'A + \omega_0^2 I & \omega_0(A' - A) \\ \omega_0(A - A') & A'A + \omega_0^2 I \end{pmatrix}.$$

Now we have to show that this matrix has the twice-repeated eigenvalues.
Consider column vectors of the same size $\underline{v_1}$, and $\underline{v_2}$.Then,

$$\begin{pmatrix} A'A + \omega_0^2 I & \omega_0(A' - A) \\ \omega_0(A - A') & A'A + \omega_0^2 I \end{pmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \lambda \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\rightarrow \begin{cases} (A'A + \omega_0^2 I)v_1 + \omega_0(A' - A)v_2 = \lambda v_1 \\ \omega_0(A - A')v_1 + (A'A + \omega_0^2 I)v_2 = \lambda v_2 \end{cases}.$$

Also, consider the following:

$$\left(\begin{array}{cc} A'A+\omega_0^2 I & \omega_0(A'-A) \\ \omega_0(A-A') & A'A+\omega_0^2 I \end{array}\right)\left[\begin{array}{c} v_2 \\ -v_1 \end{array}\right] = \lambda\left[\begin{array}{c} v_2 \\ -v_1 \end{array}\right]$$

$$\rightarrow \begin{cases} (A'A+\omega_0^2 I)v_2 - \omega_0(A'-A)v_1 & = & \lambda v_2 \\ \omega_0(A-A')v_2 - (A'A+\omega_0^2 I)v_1 & = & -\lambda v_1 \end{cases}.$$

Notice that the sets of equations above are in fact the same. This implies that there are two exactly same eigenvalues and the corresponding eigenvectors $[v_1^T\, v_2^T]^T$ and $[v_2^T - v_1^T]$ are orthogonal to each other. Thus $\lambda_{min}(A_\omega' A_\omega) = \lambda_{2^{nd}\ min}(A_\omega' A_\omega)$.
This implies that

$$\sigma_{min}(A_\omega) = \sigma_{2n}(A_\omega) = \sigma_{2n-1}(A_\omega) \because A_\omega' A_\omega \in \mathbf{R}^{2n\times 2n}.$$

Exercise 15.1 (a) The system is causal if the impulse response is right-sided. Consider a sequence $e^{-at}u[t]$, where $u[t]$ is a unit step: $u[t] = 1$ for $t \geq 0$, and zero otherwise. Laplace tranform of this sequence converges if $Re(s) > -a$, and is equal to

$$\int_{-\infty}^{\infty} e^{-st}e^{-at}u[t]dt = \frac{1}{s+a},\ \text{ROC} : Re(s) > -a$$

Therefore for a system represented by first-order transfer function to be causal the ROC has to be to the right of the pole (in fact this is true for a multiple pole as well). Since a rational function can be represented by a partial fraction expansion, and region of convergence is defined by the intersection of individual regions of convergence, the ROC of the system has to lie to the right of the rightmost pole for the system to be causal. In case of a rational transfer function this is also a sufficient condition. Note that if an LTI system has a rational transfer function, its impulse response consists of diverging or decaying exponents (maybe multiplied by powers of t), therefore all concepts of p-stability are equivalent. For BIBO stability the impulse response has to be absolutely integrable, which is equivalent to existence of Fourier transform. The Fourier transform is Laplace transform evaluated at the $j\omega$ axis. Therefore for stability the ROC has to include the $j\omega$ axis. Using these two rules we can see that the system

$$G(s) = \frac{s+2}{(s-2)(s+1)}$$

is

- (i) neither causal nor stable for ROC given by $Re(s) < -1$
- (ii) non-causal and stable for ROC $-1 < Re(s) < 2$.
- (iii) causal and unstable for ROC $Re(s) > 2$

Another way to solve the problem is to find (look up in the tables) inverse Laplace tranforms corresponding to the transfer function and ROC pairs. Compute partial fraction expansion

$$G(s) = \frac{s+2}{(s-2)(s+1)} = \frac{1}{3}\left[\frac{4}{s-2} - \frac{1}{s+1}\right]$$

The impulse response functions in three different cases are

$$\begin{aligned}
\text{(i) } h(t) &= \frac{1}{3}\left[-4e^{2t}u[-t] + e^{-t}u[-t]\right], Re(s) < -1 \text{ anticausal, unstable} \\
\text{(ii) } h(t) &= \frac{1}{3}\left[-4e^{2t}u[-t] + e^{-t}u[t]\right], -1 < Re(s) < 2 \text{ non-causal, stable} \\
\text{(iii) } h(t) &= \frac{1}{3}\left[4e^{2t}u[t] + e^{-t}u[t]\right], Re(s) > 2 \text{ causal, unstable}
\end{aligned}$$

(b) Note that if there is a diverging exponent in the impulse response, an input which is non-zero on some interval will result in an exponentially diverging output. For example, in case (iii) choose $f(t) = 1$ for $0 < t < 1$ and 0 otherwise. The output for any positive t will be a linear combination of e^{2t} and e^{-t}. For example for $t > 1$:

$$y(t) = \frac{1}{3}\int_{-\infty}^{+\infty}\left(4e^{2(t-\tau)} - e^{-(t-\tau)}\right)u[t-\tau]f(\tau)d\tau = \frac{2}{3}e^{2t}\left(1 - e^{-2}\right) - \frac{1}{3}e^{-t}(e-1)$$

Clearly this function grows unbounded and has an infinite p-norm. However the input $f(t)$ has p-norm equal to 1 for any p including ∞. In case (i) we can use $f(t) = 1$ for $-1 < t < 0$, for example. Infinitely long bounded inputs that do not cancel an unstable pole will also result in unbounded output. For example, choose $e^{-2t}, t \geq 0$ in case (iii).

Exercise 15.2 a) When $g(x) = cos(x)$, the system is unstable for $p \geq 1$. Proof: Suppose the system is p-stable. Then, there there exists a constant C such that $\|g\|_p \leq C\|u\|_p$. Now, with $\|z\|_p \to 0$, there exists a T such that $cos(z(t)) \to 1$ for all $t \geq T$. So, we have a contradiction where $\|u\|_p \to 0$, then $\|y\|_p \to 1$, which implies that there are no such a constant$C$ to satisfy the condition. Therefore the system is p-stable not all $p \geq 1$.

b) When $g(x) = sin(x)$, the system is p-stable for $p \geq 1$. Proof: Consider the Taylor series expansion of $y(t) = sin(z(t))$ about the origin. Then, we have

$$y(t) = sin(z(t)) = z(t) - \frac{1}{3}z^3(t) + H.O.T.$$

This implies that

$$\|y\|_p \leq \|z\|_p + O(\|z\|_p). \tag{1}$$

Now, because of the stability of the system from u to z , we have

$$\|z\|_p \leq C\|u\|_p \tag{2}$$

for some constant C. Thus combining Eqn (1) and (2), we have

$$\|y\|_p \leq C\|u\|_p + O(C\|u\|_p).$$

So, for all $\epsilon > 0$ there exists δ such that $O(C\|u\|_p) \leq \epsilon\|u\|_p$, which implies that $\|y\|_p \leq (C+\epsilon)\|u\|_p$. That concludes the p-stability, with $\|u\|_p < \delta$.

c) When $g(x)$ is a saturation function with a scale of 1, the system is p-stable for $p \geq 1$. Proof: Again since the system from u to z is p-stable, there exists a constant C such that $\|z\|_p \leq C\|u\|_p$. So, for all u with $\|u\|_p \leq \delta$, if we take C to be $\frac{1}{\delta}$, then we have

$$\|z\|_P \leq C\|u\|_p \leq 1.$$

Since

$$|g(z)| = \begin{cases} z & |z| \leq 1 \\ 1 & |z| \geq 1 \end{cases},$$

for $|z| \leq 1$ we have

$$\|y\|_p = \|z\|_P \leq C\|u\|_p \leq 1.$$

Therefore this system is p-stable for all $p \geq 1$ in $|z| < 1$.

Exercise 16.1 a) Since $u \in \mathbf{X}$, we can express u as

$$u = \sum_{i=1}^{N} u_i e^{j\omega_i t} \quad \text{where } u_i \in \mathbb{R}^n, \quad \omega_i \in \mathbb{R}.$$

With

$$\begin{aligned} u'(t) &= \sum_{i=1}^{N} e^{-j\omega_i t} = \sum_{i=1}^{N} u_i^T e^{-j\omega_i t} \\ \rightarrow u'(t)u(t) &= (\sum_{i=1}^{N} e^{-j\omega_i t} u_i^T)(\sum_{k=1}^{N} u_j e^{j\omega_k t}) \\ &= \sum_{i=1}^{N}\sum_{k=1}^{N} e^{j(\omega_k - \omega_i)t} u_i^T u_k, \end{aligned}$$

we can compute P_u as follows:

$$\begin{aligned} P_u &= \lim_{L\to\infty}\left(\frac{1}{2L}\int_{-L}^{L} u'(t)u(t)dt\right) \\ &= \lim_{L\to\infty}\frac{1}{2L}\int_{-L}^{L}\sum_i\sum_j u_i^T u_j e^{j(\omega_j-\omega_i)t}dt \\ &= \lim_{L\to\infty}\frac{1}{2L}\sum_i\sum_j u_i^T u_j \int_{-L}^{L} e^{j(\omega_j-\omega_i)t}dt. \end{aligned}$$

Note that as $L \to \infty$, because of the orthnormality of complex exponential,

$$\lim_{L\to\infty}\int_{-L}^{L} e^{j(\omega_j-\omega_i)tdt} = \begin{cases} 0 & : \quad i \neq j \\ 1 & : \quad i = j \end{cases}$$

Thus

$$P_u = \lim_{L\to\infty}\frac{1}{2L}\sum_{i=1}^{N} u_i^T u_i(2L) = \sum_{i=1}^{N}\|u_i\|_2^2.$$

b) The output of the system can be expressed as $\underline{y}(t) = \mathcal{H}(t) * \underline{u}(t)$ in time domain or $Y(s) = H(s)U(s)$ in frequency domain. For a CT LTI system, we have $y = H(j\omega_i)u_i e^{j\omega_i t}$ if $u = u_i e^{j\omega_i t}$. Thus

$$y(t) = \sum_{i=1}^{N} H(j\omega_i)u_i e^{j\omega_i t}.$$

Following the similar method taken in a), we have

$$\begin{aligned}y'(t)y(t) &= \sum_{i=1}^{N} u_i^T H'(j\omega_i)e^{-j\omega_i t}\sum_{k=1}^{N} H(j\omega_k)u_k e^{j\omega_k t} \\ &= \sum_{i=1}^{N}\sum_{k=1}^{N} e^{j(\omega_k-\omega_i)t}u_i^T H'(j\omega_i)H(j\omega_k)u_k.\end{aligned}$$

Thus, P_y can be computed as follows:

$$\begin{aligned}P_y &= \lim_{L\to\infty}\frac{1}{2L}\sum_i\sum_k u_i^T H'(j\omega_i)H(j\omega_k)u_k\int_{-L}^{L} e^{j(\omega_k-\omega_i)t}dt \\ &= \lim_{L\to\infty}\frac{1}{2L}\sum_i \|H(j\omega_i)u_i\|^2(2L). \\ \therefore P_y &= \sum_{i=1}^{N}\|H(j\omega_i)u_i\|^2.\end{aligned}$$

c)Using the fact shown in b),

$$\begin{aligned}P_y &= \sum_{i=1}^{N}\|H(j\omega_i)u_i\|^2 \\ &\le \sum_{i=1}^{N}\sigma_{max}^2(H(j\omega_i))\|u_i\|^2 \\ &\le \max_i \sigma_{max}^2(H(j\omega_i))\sum_{i=1}^{N}\|u_i\|^2 \\ &= \max_i \sigma_{max}^2(H(j\omega_i))P_u \\ \to P_y &\le \max_i \sigma_{max}^2(H(j\omega_i))P_u \\ P_y &\le \sup_\omega \sigma_{max}^2(H(j\omega))P_u. \\ \therefore \sup_{P_u=1} P_y &= \|H\|_\infty^2.\end{aligned}$$

d)Now we have to find an input $u \in \mathbf{X}$ such that $P_y = \|H\|_\infty^2 P_u$. Consider a SVD of $H(j\omega_0)$:

$$H(j\omega_0) = U\Sigma V' = \begin{pmatrix} | & & | \\ u_1 & \cdots & u_n \\ | & & | \end{pmatrix}\begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{pmatrix}\begin{pmatrix} - & v_1' & - \\ & \vdots & \\ - & v_n' & - \end{pmatrix}.$$

Let $u = v_1 e^{j\omega_0}$ where ω_0 is such that $\|H\|_\infty = \sigma_{max}(H(j\omega_0))$, then

$$\begin{aligned}P_y &= \|H(j\omega_0)v_1 e^{j\omega_0}\|_2^2 = \|H(j\omega_0)v_1\|_2^2 = \sigma_1^2\|u_1\|_2^2. \\ \therefore P_y &= \sigma_{max}^2(H(j\omega_0)).\end{aligned}$$

Indeed, the equality can be achieved by the choice of $u = v_1 e^{j\omega_0}$.

Exercise 16.2 (a) Consider a SISO DT system; we have that $y(n) = \sum_{k=-\infty}^{\infty} h(k-n)x(k)$. So,

$$\begin{aligned}
\|y\|_2^2 &= \sum_{n=-\infty}^{\infty} y^2(n) \\
&= \sum_{n=-\infty}^{\infty} (\sum_{k=-\infty}^{\infty} h(k-n)x(k))^2 \\
&\leq \sum_{n=-\infty}^{\infty} (\sum_{k=-\infty}^{\infty} h(k-n)^{\frac{1}{2}} h(k-n)^{\frac{1}{2}} x(k))^2 \\
&\leq \sum_{n=-\infty}^{\infty} (\sum_{k=-\infty}^{\infty} |h(k-n)|) \sum_{k=-\infty}^{\infty} (|h(k-n)|x^2(k))
\end{aligned}$$

where the inequality follows by the Cauchy inequality, $\left(\sum a_k b_k\right)^2 \leq \left(\sum a_k^2\right)\left(\sum b_k^2\right)$, where $a_k = h(k-n)^{\frac{1}{2}}$ and $b_k = h(k-n)^{\frac{1}{2}} x(k)$. Now note that $\sum_{k=-\infty}^{\infty} |h(k-n)| = \|h\|_1$. So,

$$\begin{aligned}
\|y\|_2^2 &\leq \|h\|_1 \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} |h(k-n)|x^2(k) \\
&\leq \|h\|_1 \sum_{k=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} |h(k-n)|x^2(k) \\
&\leq \|h\|_1 \sum_{k=-\infty}^{\infty} x^2(k) \sum_{n=-\infty}^{\infty} |h(k-n)| \\
&\leq \|h\|_1^2 \sum_{k=-\infty}^{\infty} x^2(k) \\
&\leq \|h\|_1^2 \|x\|_2^2.
\end{aligned}$$

(b) Suppose that h is $\mathcal{L}_1$-stable, i.e. $\sum_{k=-\infty}^{\infty} |h(k)| = M < \infty$. Now, $H(e^{j\theta}) = \sum_{k=-\infty}^{\infty} h(k)e^{-jk\theta}$. To show continuity of $H(e^{j\theta})$, we need to show that for every $\epsilon > 0$ there exists some δ such that if $|\theta - \theta_0| < \delta$ then $|H(e^{j\theta}) - H(e^{j\theta_0})| < \epsilon$. Now,

$$\begin{aligned}
|H(e^{j\theta}) - H(e^{j\theta_0})| &= |\sum_{k=-\infty}^{\infty} h(k)(e^{-j\theta k} - e^{-j\theta_0 k})| \\
&\leq \sum_{k=-\infty}^{\infty} |h(k)||e^{-j\theta k} - e^{-j\theta_0 k}| \\
&\leq |e^{-j\theta k_0} - e^{-j\theta_0 k_0}| \sum_{k=-\infty}^{\infty} |h(k)| \\
&\leq |e^{-j\theta k_0} - e^{-j\theta_0 k_0}| M
\end{aligned}$$

where k_0 is the index for which $|e^{-j\theta k} - e^{-j\theta_0 k}|$ is maximum. Now, $e^{-j\theta}$ is continuous (since $e^{-j\theta} = \cos\theta - j\sin\theta$), so given ϵ/M, there is a δ such that $|\theta - \theta_0| < \delta$ implies $|e^{-j\theta k} - e^{-j\theta_0 k}| < \epsilon/M$, which in turn implies that $|H(e^{j\theta}) - H(e^{j\theta_0})| < \epsilon$.

First, we note that e^z is analytic (i.e. it is continuous and its derivative exists) everywhere on $\mathbb{C}$. Furthermore, $\frac{1}{z^{-1}-1} = \frac{z}{1-z}$ is analytic everywhere except at 1 (is it the quotient of two analytic functions, z and $1-z$). So, $e^{\frac{1}{z^{-1}-1}}$ is analytic on the region of overlap since it is the composition of two analytic functions and the composition of analytic functions is analytic. At the point $z = 1$, however, this function has an essential singularity, and its limit does not exist. For example, consider the behavior of the function as we aproach it through the real axis (i.e. $z = x$, $x \in \mathbb{R}$) $lim_{x\to 1} e^{\frac{x}{1-x}} = 0$ as x tends to 1 from the right, whereas $lim_{x\to 1} e^{\frac{x}{1-x}} = \infty$ as x tends to 1 from the left. Now, on the unit circle, $z = e^{j\theta} = \cos\theta + j\sin\theta$. Substituting $z = e^{j\theta}$ in $H(z)$, we have that, after some simplifications,

$$H(e^{j\theta}) = e^{-\frac{1}{2}+j\frac{\sin\theta}{2(1-\cos\theta)}},$$

from which we have that $|H(e^{j\theta})| = e^{-1/2}$.