LECTURE 20

LECTURE OUTLINE

- The primal function
- Conditions for strong duality
- Sensitivity
- Fritz John conditions for convex programming

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- Problem: Minimize $f(x)$ subject to $x \in X$, and $g_1(x) \leq 0, \ldots, g_r(x) \leq 0$ (assuming $-\infty < f^* < \infty$). It is equivalent to $\inf_{x\in X} \sup_{\mu\geq 0} L(x, \mu)$.

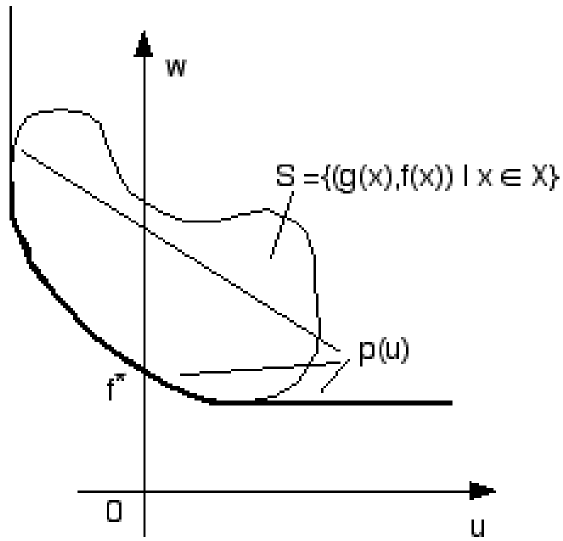
- The primal function is the *perturbed optimal value*

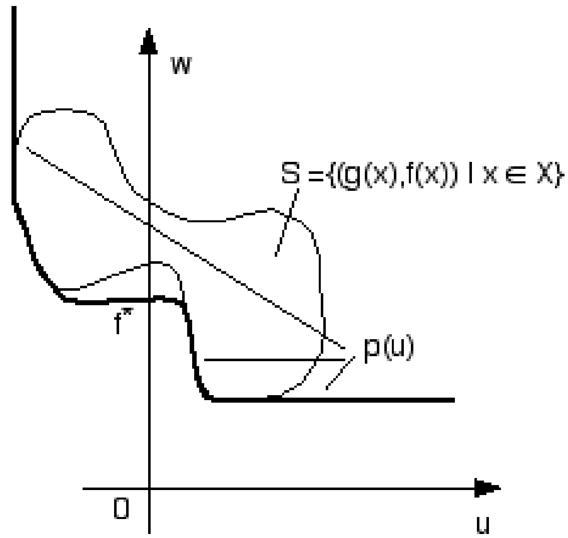
$$p(u) = \inf_{x\in X} \sup_{\mu\geq 0} \big\{L(x, \mu) - \mu' u\big\} = \inf_{\substack{x\in X\\ g(x)\leq u}} f(x)$$

- Note that $p(u)$ is the result of partial minimization over X of the function $F(x, u)$ given by

$$F(x, u) = \begin{cases} f(x) & \text{if } x \in X \text{ and } g(x) \leq u, \\ \infty & \text{otherwise.} \end{cases}$$

PRIMAL FUNCTION AND STRONG DUALITY





- Apply min common-max crossing framework with set $M = \text{epi}(p)$, assuming p is convex and $-\infty < p(0) < \infty$.

- There is no duality gap if and only if p is lower semicontinuous at $u = 0$.

- Conditions that guarantee lower semicontinuity at $u = 0$, correspond to those for preservation of closure under the partial minimization $p(u) = \inf_{\substack{x \in X \\ g(x) \le u}} f(x)$, e.g.:
  - X is convex and compact, f, g_j: convex.
  - Extensions involving the recession cones of X, f, g_j.
  - $X = \Re^n$, f, g_j: convex quadratic.

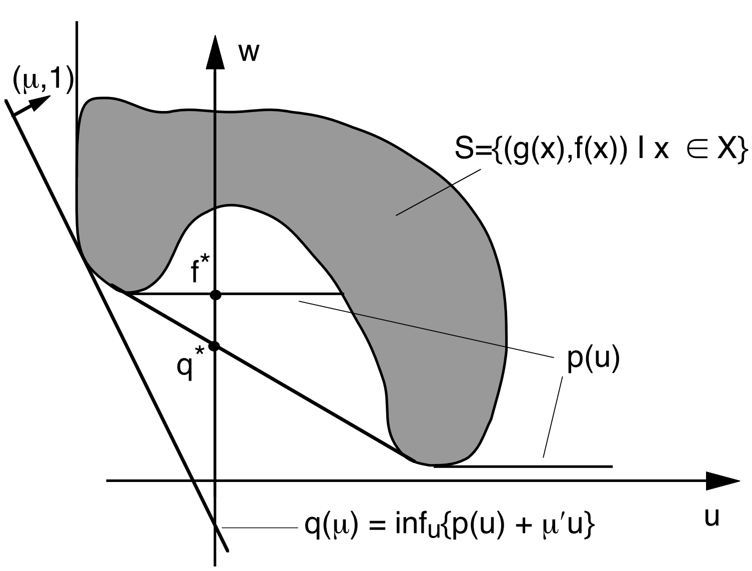
RELATION OF PRIMAL AND DUAL FUNCTIONS

- Consider the dual function q. For every $\mu \geq 0$, we have

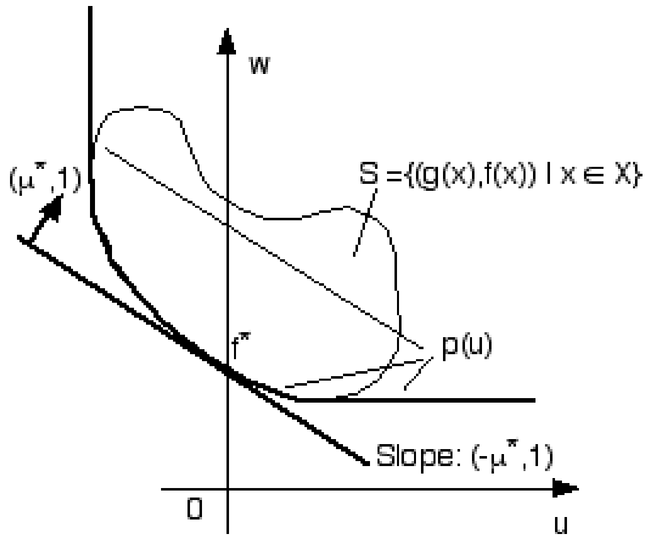
$$
\begin{aligned}
q(\mu) &= \inf_{x \in X} \{f(x) + \mu' g(x)\} \\
&= \inf_{\{(u,x) \mid x \in X,\, g(x) \leq u,\, j=1,\ldots,r\}} \{f(x) + \mu' g(x)\} \\
&= \inf_{\{(u,x) \mid x \in X,\, g(x) \leq u\}} \{f(x) + \mu' u\} \\
&= \inf_{u \in \Re^r} \inf_{x \in X,\, g(x) \leq u} \{f(x) + \mu' u\}.
\end{aligned}
$$

- Thus

$$q(\mu) = \inf_{u \in \Re^r} \big\{p(u) + \mu' u\big\}, \qquad \forall\, \mu \geq 0,$$



SUBGRADIENTS OF THE PRIMAL FUNCTION



- Assume that p is convex, $p(0)$ is finite, and p is proper. Then:
  - The set of G-multipliers is $-\partial p(0)$ (negative subdifferential of p at $u = 0$). This follows from the relation

$$q(\mu) = \inf_{u \in \Re^r} \{p(u) + \mu' u\}.$$

  - If the origin lies in the relative interior of the effective domain of p, then there exists a G-multiplier.
  - If the origin lies in the interior of the effective domain of p, the set of G-multipliers is nonempty and compact.

SENSITIVITY ANALYSIS I

- Assume that p is convex and differentiable. Then $-\nabla p(0)$ is the unique G-multiplier μ^*, and we have

$$\mu_j^* = -\frac{\partial p(0)}{\partial u_j}, \qquad \forall\ j.$$

- Let μ^* be a G-multiplier, and consider a vector u_j^γ of the form

$$u_j^\gamma = (0, \ldots, 0, \gamma, 0, \ldots, 0)$$

where γ is a scalar in the jth position. Then

$$\lim_{\gamma \uparrow 0} \frac{p(u_j^\gamma) - p(0)}{\gamma} \le -\mu_j^* \le \lim_{\gamma \downarrow 0} \frac{p(u_j^\gamma) - p(0)}{\gamma}.$$

Thus $-\mu_j^*$ lies between the left and the right slope of p in the direction of the jth axis starting at $u = 0$.

SENSITIVITY ANALYSIS II

- Assume that p is convex and finite in a neighborhood of 0. Then, from the theory of subgradients:
  - $\partial p(0)$ is nonempty and compact.
  - The directional derivative $p'(0; y)$ is a real-valued convex function of y satisfying
$$p'(0; y) = \max_{g \in \partial p(0)} y'g$$

- Consider the direction of steepest descent of p at 0, i.e., the $\overline{y}$ that minimizes $p'(0; y)$ over $\|y\| \leq 1$. Using the Saddle Point Theorem,

$$p'(0; \overline{y}) = \min_{\|y\| \leq 1} p'(0; y) = \min_{\|y\| \leq 1} \max_{g \in \partial p(0)} y'g = \max_{g \in \partial p(0)} \min_{\|y\| \leq 1} y'g$$

- The saddle point is $(g^*, \overline{y})$, where g^* is the subgradient of minimum norm in $\partial p(0)$ and $\overline{y} = -g^*/\|g^*\|$. The min-max value is $-\|g^*\|$.

- **Conclusion:** If μ^* is the G-multiplier of minimum norm and $\mu^* \neq 0$, the direction of steepest descent of p at 0 is $\overline{y} = \mu^*/\|\mu^*\|$, while the rate of steepest descent (per unit norm of constraint violation) is $\|\mu^*\|$.

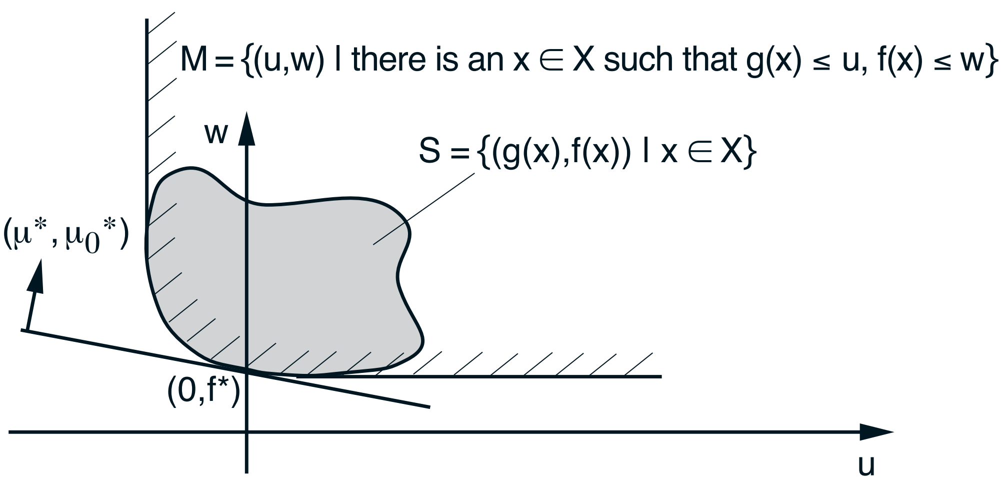
FRITZ JOHN THEORY FOR CONVEX PROBLEMS

• Assume that X is convex, the functions f and g_j are convex over X, and $f^* < \infty$. Then there exist a scalar μ_0^* and a vector $\mu^* = (\mu_1^*, \ldots, \mu_r^*)$ satisfying the following conditions:

(i) $\mu_0^* f^* = \inf_{x \in X} \big\{ \mu_0^* f(x) + \mu^{*\prime} g(x) \big\}$.

(ii) $\mu_j^* \geq 0$ for all $j = 0, 1, \ldots, r$.

(iii) $\mu_0^*, \mu_1^*, \ldots, \mu_r^*$ are not all equal to 0.



• If the multiplier μ_0^* can be proved positive, then μ^*/μ_0^* is a G-multiplier.

• Under the Slater condition (there exists $\overline{x} \in X$ s.t. $g(\overline{x}) < 0$), μ_0^* cannot be 0; if it were, then $0 = \inf_{x \in X} \mu^{*\prime} g(x)$ for some $\mu^* \geq 0$ with $\mu^* \neq 0$, while we would also have $\mu^{*\prime} g(\overline{x}) < 0$.

FRITZ JOHN THEORY FOR LINEAR CONSTRAINTS

- Assume that X is convex, f is convex over X, the g_j are affine, and $f^* < \infty$. Then there exist a scalar μ_0^* and a vector $\mu^* = (\mu_1^*, \ldots, \mu_r^*)$, satisfying the following conditions:

  (i) $\mu_0^* f^* = \inf_{x \in X} \big\{ \mu_0^* f(x) + \mu^{*\prime} g(x) \big\}$.

  (ii) $\mu_j^* \geq 0$ for all $j = 0, 1, \ldots, r$.

  (iii) $\mu_0^*, \mu_1^*, \ldots, \mu_r^*$ are not all equal to 0.

  (iv) If the index set $J = \{j \neq 0 \mid \mu_j^* > 0\}$ is nonempty, there exists a vector $\tilde{x} \in X$ such that $f(\tilde{x}) < f^*$ and $\mu^{*\prime} g(\tilde{x}) > 0$.

- Proof uses Polyhedral Proper Separation Th.

- Can be used to show that there exists a geometric multiplier if $X = P \cap C$, where P is polyhedral, and $\mathrm{ri}(C)$ contains a feasible solution.

- **Conclusion:** The Fritz John theory is sufficiently powerful to show the major constraint qualification theorems for convex programming.

- There is more material on pseudonormality, informative geometric multipliers, etc.