LECTURE 6

LECTURE OUTLINE

- Nonemptiness of closed set intersections
- Existence of optimal solutions
- Special cases: Linear and quadratic programs
- Preservation of closure under linear transformation and partial minimization

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Let

$$X^* = \arg\min_{x \in X} f(x), \qquad f, X\text{: closed convex}$$

Then X^* is nonempty and compact iff X and f have no common nonzero direction of recession.

The proof is based on a set intersection argument: Let $f^* = \inf_{x \in \Re^n} f(x)$, let $\{\gamma_k\}$ be a scalar sequence such that $\gamma_k \downarrow f^*$, and note that

$$X^* = \cap_{k=0}^{\infty} \{x \in X \mid f(x) \le \gamma_k\}$$

THE ROLE OF CLOSED SET INTERSECTIONS

• Given a sequence of nonempty closed sets $\{S_k\}$ in $\Re^n$ with $S_{k+1} \subset S_k$ for all k, when is $\cap_{k=0}^{\infty} S_k$ nonempty?

• Set intersection theorems are significant in at least three major contexts, which we will discuss in what follows:

• Does a function $f : \Re^n \mapsto (-\infty, \infty]$ attain a minimum over a set X? This is true iff the intersection of the nonempty level sets $\{x \in X \mid f(x) \leq \gamma_k\}$ is nonempty.

• If C is closed and A is a matrix, is $A\,C$ closed? Special case:

- If C_1 and C_2 are closed, is $C_1 + C_2$ closed?

• If $F(x, z)$ is closed, is $f(x) = \inf_z F(x, z)$ closed? (Critical question in duality theory.) Can be addressed by using the relation

$$P\big(\mathrm{epi}(F)\big) \subset \mathrm{epi}(f) \subset \mathrm{cl}\Big(P\big(\mathrm{epi}(F)\big)\Big).$$

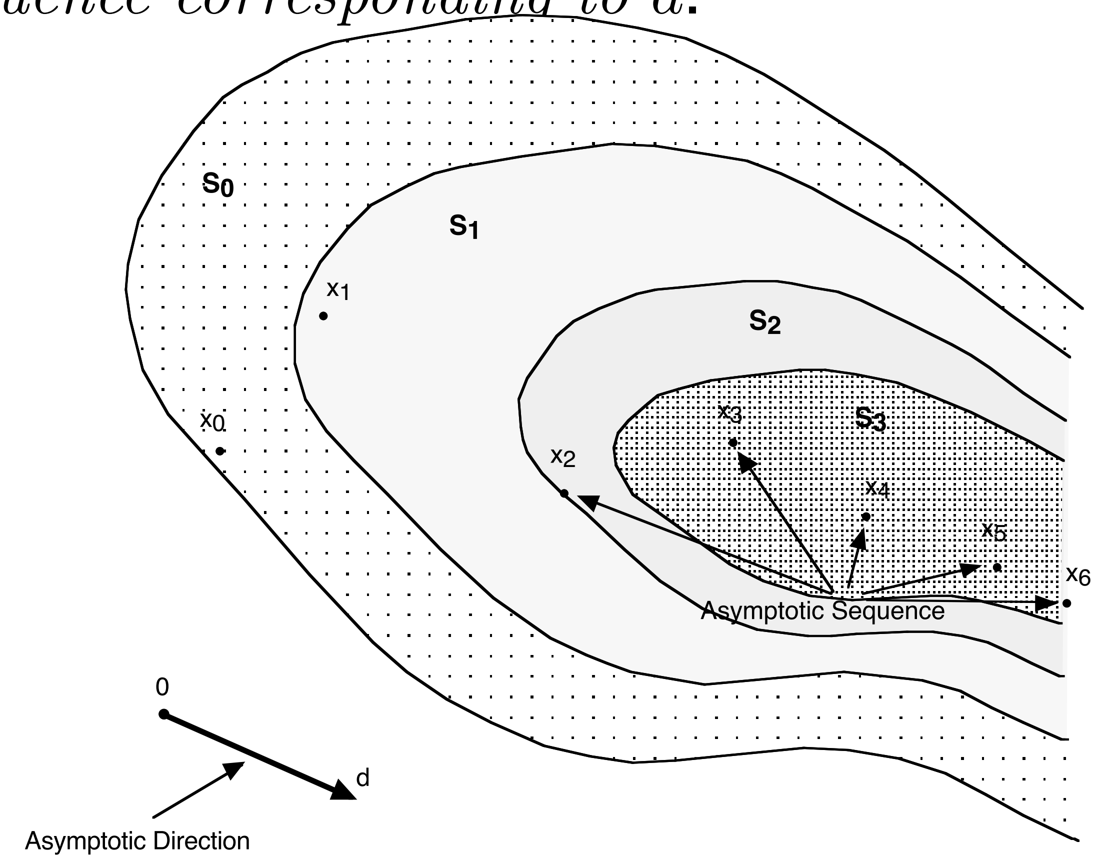
ASYMPTOTIC DIRECTIONS

- Given a sequence of nonempty nested closed sets $\{S_k\}$, we say that a vector $d \neq 0$ is an *asymptotic direction of* $\{S_k\}$ if there exists $\{x_k\}$ s. t.

$$x_k \in S_k, \qquad x_k \neq 0, \qquad k = 0, 1, \ldots$$

$$\|x_k\| \to \infty, \qquad \frac{x_k}{\|x_k\|} \to \frac{d}{\|d\|}.$$

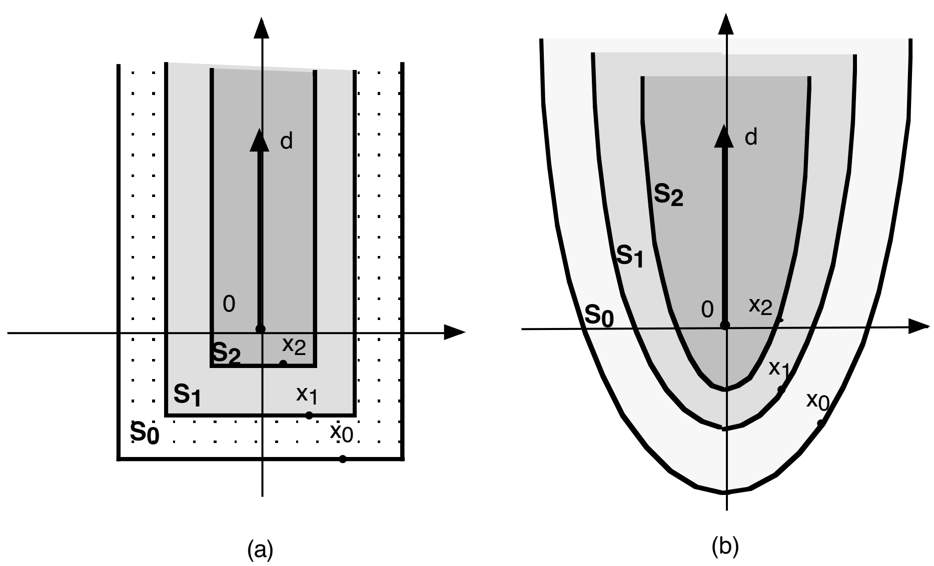
- A sequence $\{x_k\}$ associated with an asymptotic direction d as above is called an *asymptotic sequence corresponding to* d.



RETRACTIVE ASYMPTOTIC DIRECTIONS

- An asymptotic sequence $\{x_k\}$ is called *retractive* if there exists a bounded positive sequence $\{\alpha_k\}$ and an index $\overline{k}$ such that

$$x_k - \alpha_k d \in S_k, \qquad \forall\, k \geq \overline{k}.$$



- Important observation: A retractive asymptotic sequence $\{x_k\}$ (for large k) gets closer to 0 when shifted in the opposite direction $-d$.

SET INTERSECTION THEOREM

- If all asymptotic sequences corresponding to asymptotic directions of $\{S_k\}$ are retractive. Then $\cap_{k=0}^{\infty} S_k$ is nonempty.
- Key proof ideas:
  (a) The intersection $\cap_{k=0}^{\infty} S_k$ is empty iff there is an unbounded sequence $\{x_k\}$ consisting of minimum norm vectors from the S_k.

  (b) An asymptotic sequence $\{x_k\}$ consisting of minimum norm vectors from the S_k cannot be retractive, because such a sequence eventually gets closer to 0 when shifted opposite to the asymptotic direction.

CONNECTION WITH RECESSION CONES

- The asymptotic directions of a closed *convex* set sequence $\{S_k\}$ are the nonzero vectors in the intersection of the recession cones $\cap_{k=0}^{\infty} R_{C_k}$.
- Asymptotic directions that are also lineality directions are retractive.
- Apply the intersection theorem for the case of convex S_k: $\cap_{k=0}^{\infty} S_k$ is nonempty if $R = L$, where

$$R = \cap_{k=0}^{\infty} R_{S_k}, \qquad L = \cap_{k=0}^{\infty} L_{S_k}$$

- We say that d is an *asymptotic direction of a nonempty closed set* S if it is an asymptotic direction of the sequence $\{S_k\}$, where $S_k = S$ for all k.
- The asymptotic directions of a closed convex set S are the nonzero vectors in the recession cone R_S.
- The asymptotic sequences of polyhedral sets of the form $X = \{x \mid a_j' x + b_j \leq 0,\ j = 1, \ldots, r\}$ are retractive.

SET INTERSECTION WITH POLYHEDRAL SETS

- Set Intersection Theorem for Partially Polyhedral Sets: Let X be polyhedral of the form $X = \{x \mid a_j'x + b_j \leq 0,\ j = 1, \ldots, r\}$ and let

$$S_k = X \cap \overline{S}_k$$

If all the asymptotic directions d of $\{\overline{S}_k\}$ that satisfy $a_j'd \leq 0$ for all $j = 1, \ldots, r$, have asymptotic sequences that are retractive, then $\cap_{k=0}^{\infty} S_k$ is nonempty.

- Proof idea: The asymptotic sequences of $\{S_k\}$ must be asymptotic for X and for $\{S_k\}$.

Applying the intersection theorem to $\overline{S}_k$, we have that $\cap_{k=0}^{\infty} S_k$ is nonempty if

$$R_X \cap R \subset L,$$

where

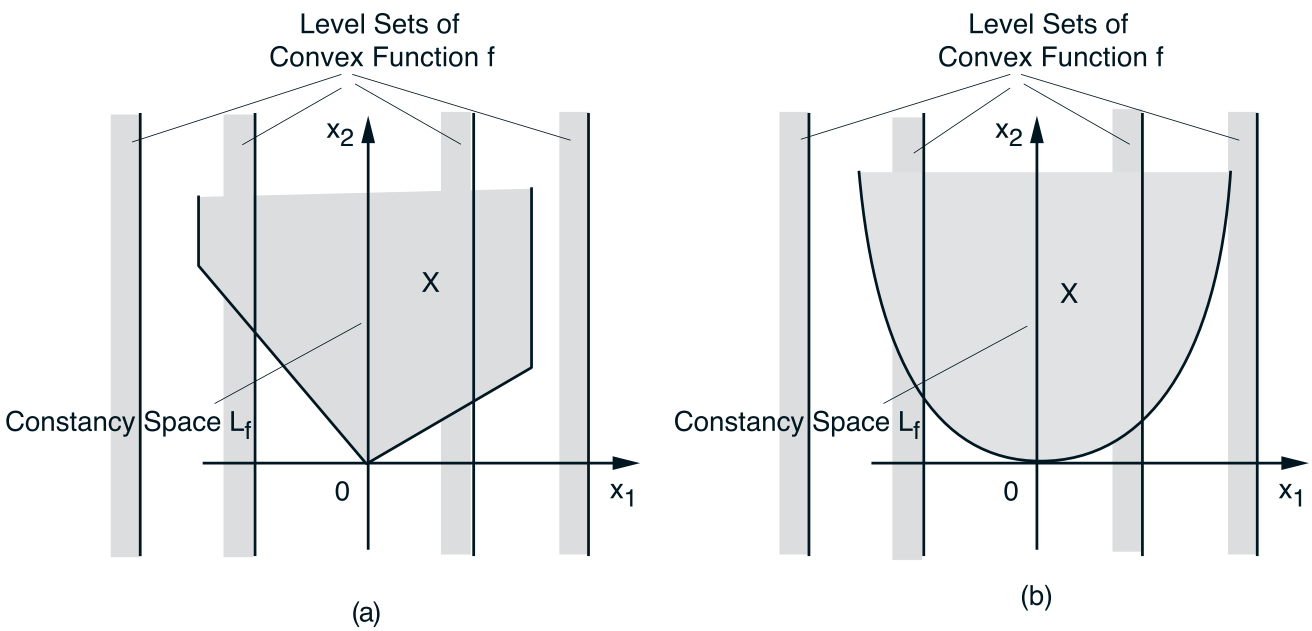
$$R = \cap_{k=0}^{\infty} R_{\overline{S}_k}, \qquad L = \cap_{k=0}^{\infty} L_{\overline{S}_k}$$

EXISTENCE OF OPTIMAL SOLUTIONS

- Let X and $f : \Re^n \mapsto (-\infty, \infty]$ be closed convex and such that $X \cap \mathrm{dom}(f) \neq \emptyset$. The set of minima of f over X is nonempty under any one of the following two conditions:

  (1) $R_X \cap R_f = L_X \cap L_f$.

  (2) $R_X \cap R_f \subset L_f$, and X is polyhedral.



Example: Here $f(x_1, x_2) = e^{x_1}$. In (a), X is polyhedral, and the minimum is attained. In (b),

$$f(x_1, x_2) = e^{x_1}, \qquad X = \big\{(x_1, x_2) \mid x_1^2 \leq x_2\big\}.$$

We have $R_X \cap R_f \subset L_f$, but the minimum is not attained (X is not polyhedral).

LINEAR AND QUADRATIC PROGRAMMING

- Frank-Wolfe Theorem: Let

$$f(x) = x'Qx+c'x, \;\; X = \{x \mid a_j'x+b_j \leq 0,\; j = 1, \ldots, r\},$$

where Q is symmetric (not necessarily positive semidefinite). If the minimal value of f over X is finite, there exists a minimum of f of over X.

- Proof (outline): Choose $\{\gamma_k\}$ s.t. $\gamma_k \downarrow f^*$, where f^* is the optimal value, and let

$$S_k = \{x \in X \mid x'Qx + c'x \leq \gamma_k\}.$$

The set of optimal solutions is $\cap_{k=0}^{\infty} S_k$, so it will suffice to show that for each asymptotic direction of $\{S_k\}$, each corresponding asymptotic sequence is retractive.

PROOF OUTLINE – CONTINUED

- Choose an asymptotic direction d and a corresponding asymptotic direction.
- First show that

$$d'Qd \le 0, \qquad a_j'd \le 0, \quad j = 1, \ldots, r.$$

Then show that

$$(c + 2Qx)'d \ge 0, \qquad \forall\, x \in X.$$

- Then argue that for any fixed $\alpha > 0$, and k sufficiently large, we have $x_k - \alpha d \in X$. Furthermore,

$$\begin{aligned} f(x_k - \alpha d) &= (x_k - \alpha d)'Q(x_k - \alpha d) + c'(x_k - \alpha d) \\ &= x_k{}'Qx_k + c'x_k - \alpha(c + 2Qx_k)'d + \alpha^2 d'Qd \\ &\le x_k{}'Qx_k + c'x_k \\ &\le \gamma_k, \end{aligned}$$

so $x_k - \alpha d \in S_k$. **Q.E.D.**

CLOSURE UNDER LINEAR TRANSFORMATIONS

- Let C be a nonempty closed convex, and let A be a matrix with nullspace $N(A)$.

  (a) If $R_C \cap N(A) \subset L_C$, then the set $A\,C$ is closed.

  (b) Let X be a polyhedral set. If

$$R_X \cap R_C \cap N(A) \subset L_C,$$

then the set $A(X \cap C)$ is closed.

- Proof (outline): Let $\{y_k\} \subset A\,C$ with $y_k \to \overline{y}$. We prove that $\cap_{k=0}^{\infty} S_k \neq \emptyset$, where $S_k = C \cap N_k$, and

$$N_k = \{x \mid Ax \in W_k\}, \quad W_k = \big\{z \mid \|z - \overline{y}\| \leq \|y_k - \overline{y}\|\big\}$$