LECTURE 11

LECTURE OUTLINE

- Extreme points
- Extreme points of polyhedral sets
- Extreme points and linear/integer programming

Recall some of the facts of polyhedral convexity:

- Polarity relation between polyhedral and finitely generated cones

$$\{x \mid a_j'x \le 0,\ j = 1, \ldots, r\} = \mathrm{cone}\big(\{a_1, \ldots, a_r\}\big)^*.$$

- Farkas' Lemma

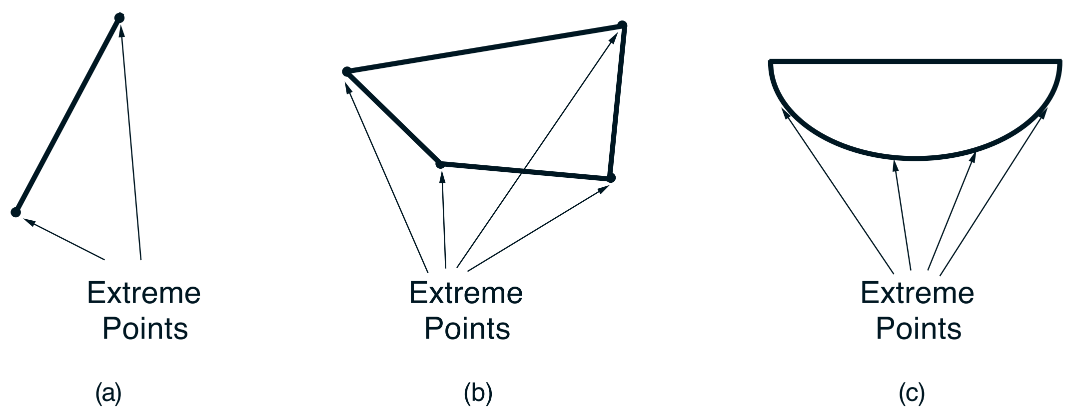
$$\{x \mid a_j'x \le 0,\ j = 1, \ldots, r\}^* = \mathrm{cone}\big(\{a_1, \ldots, a_r\}\big).$$

- Minkowski-Weyl Theorem: a cone is polyhedral iff it is finitely generated. A corollary (essentially):

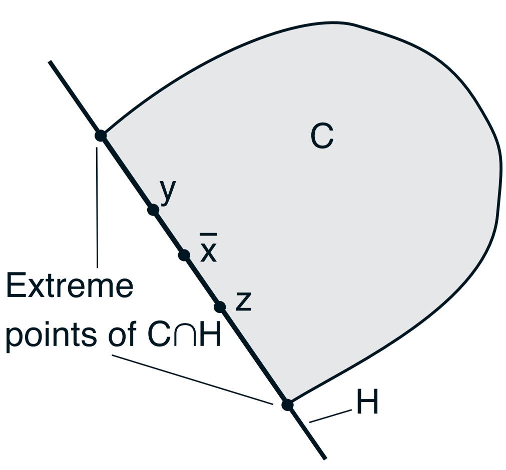
Polyhedral set $P = \mathrm{conv}\big(\{v_1, \ldots, v_m\}\big) + R_P$

EXTREME POINTS

- A vector x is an *extreme point* of a convex set C if $x \in C$ and x cannot be expressed as a convex combination of two vectors of C, both of which are different from x.



Proposition: Let C be closed and convex. If H is a hyperplane that contains C in one of its closed halfspaces, then every extreme point of $C \cap H$ is also an extreme point of C.



Proof: Let $\overline{x} \in C \cap H$ be a nonextreme point of C. Then $\overline{x} = \alpha y + (1-\alpha)z$ for some $\alpha \in (0,1)$, $y, z \in C$, with $y \neq x$ and $z \neq x$. Since $\overline{x} \in H$, the closed halfspace containing C is of the form $\{x \mid a'x \geq a'\overline{x}\}$. Then $a'y \geq a'\overline{x}$ and $a'z \geq a'\overline{x}$, which in view of $\overline{x} = \alpha y + (1-\alpha)z$, implies that $a'y = a'\overline{x}$ and $a'z = a'\overline{x}$. Thus, $y, z \in C \cap H$, showing that $\overline{x}$ is not an extreme point of $C \cap H$.

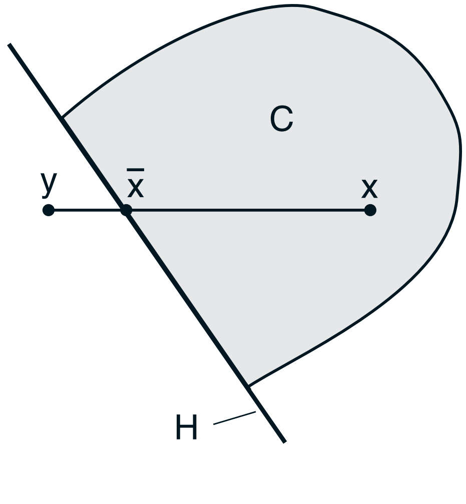
PROPERTIES OF EXTREME POINTS I

Proposition: A closed and convex set has at least one extreme point if and only if it does not contain a line.

Proof: If C contains a line, then this line translated to pass through an extreme point is fully contained in C - impossible.

Conversely, we use induction on the dimension of the space to show that if C does not contain a line, it must have an extreme point. True in $\Re$, so assume it is true in $\Re^{n-1}$, where $n \geq 2$. We will show it is true in $\Re^n$.

Since C does not contain a line, there must exist points $x \in C$ and $y \notin C$. Consider the relative boundary point $\overline{x}$.



The set $C \cap H$ lies in an $(n-1)$-dimensional space and does not contain a line, so it contains an extreme point. By the preceding proposition, this extreme point must also be an extreme point of C.

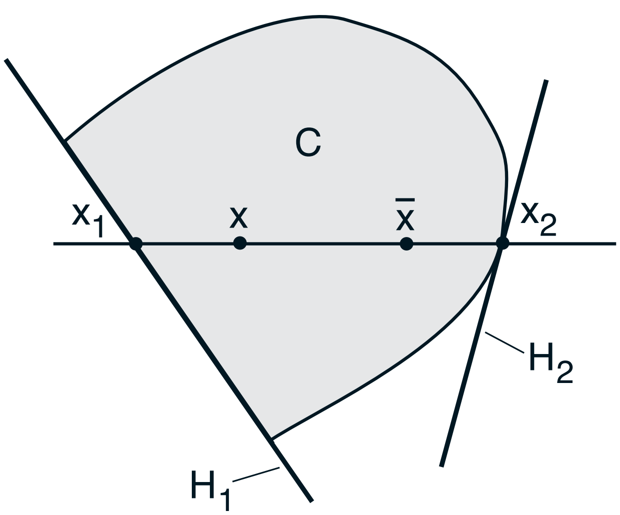
PROPERTIES OF EXTREME POINTS II

Krein-Milman Theorem: A convex and compact set is equal to the convex hull of its extreme points.

Proof: By convexity, the given set contains the convex hull of its extreme points.

Next show the reverse, i.e, every x in a compact and convex set C can be represented as a convex combination of extreme points of C.

Use induction on the dimension of the space. The result is true in $\Re$. Assume it is true for all convex and compact sets in $\Re^{n-1}$. Let $C \subset \Re^n$ and $x \in C$.



If $\overline{x}$ is another point in C, the points x_1 and x_2 shown can be represented as convex combinations of extreme points of the lower dimensional convex and compact sets $C \cap H_1$ and $C \cap H_2$, which are also extreme points of C.

EXTREME POINTS OF POLYHEDRAL SETS

• Let P be a polyhedral subset of $\Re^n$. If the set of extreme points of P is nonempty, then it is finite.

Proof: Consider the representation $P = \hat{P} + C$, where

$$\hat{P} = \mathrm{conv}\big(\{v_1, \ldots, v_m\}\big)$$

and C is a finitely generated cone.

• An extreme point $\overline{x}$ of P cannot be of the form $\overline{x} = \hat{x} + y$, where $\hat{x} \in \hat{P}$ and $y \neq 0$, $y \in C$, since in this case $\overline{x}$ would be the midpoint of the line segment connecting the distinct vectors $\hat{x}$ and $\hat{x} + 2y$. Therefore, an extreme point of P must belong to $\hat{P}$, and since $\hat{P} \subset P$, it must also be an extreme point of $\hat{P}$.

• An extreme point of $\hat{P}$ must be one of the vectors $v_1, \ldots, v_m$, since otherwise this point would be expressible as a convex combination of $v_1, \ldots, v_m$. Thus the extreme points of P belong to the finite set $\{v_1, \ldots, v_m\}$. **Q.E.D.**

CHARACTERIZATION OF EXTREME POINTS

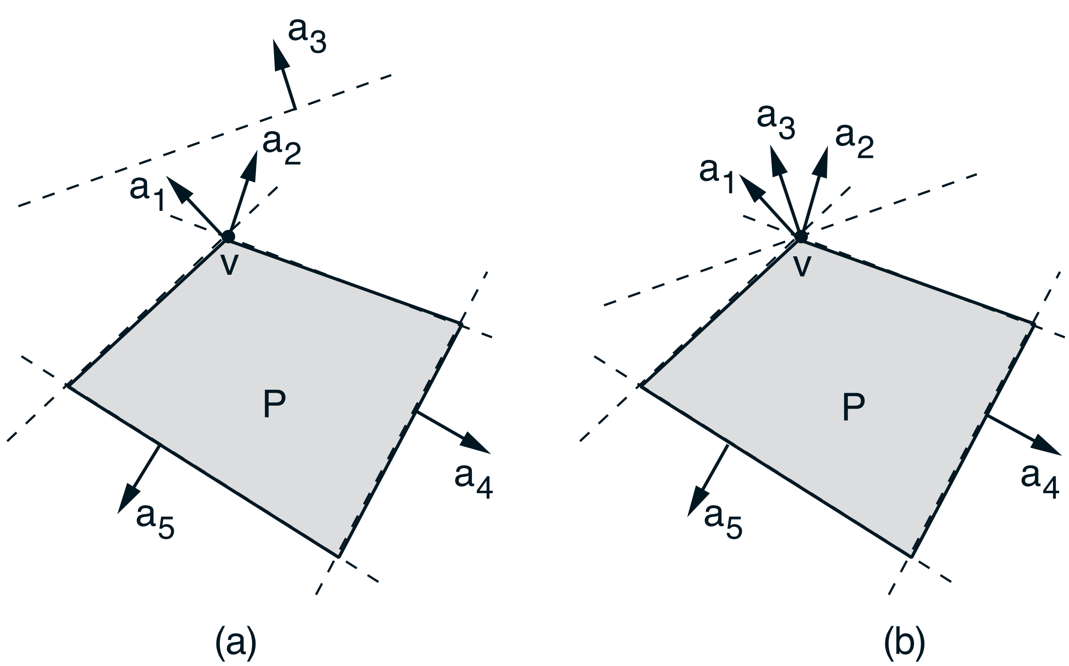
Proposition: Let P be a polyhedral subset of $\Re^n$. If P has the form

$$P = \{x \mid a_j'x \le b_j,\ j = 1, \ldots, r\},$$

where a_j and b_j are given vectors and scalars, respectively, then a vector $v \in P$ is an extreme point of P if and only if the set

$$A_v = \{a_j \mid a_j'v = b_j,\ j \in \{1, \ldots, r\}\}$$

contains n linearly independent vectors.



PROOF OUTLINE

If the set A_v contains fewer than n linearly independent vectors, then the system of equations

$$a_j' w = 0, \qquad \forall\, a_j \in A_v$$

has a nonzero solution $\overline{w}$. For small $\gamma > 0$, we have $v + \gamma\overline{w} \in P$ and $v - \gamma\overline{w} \in P$, thus showing that v is not extreme. Thus, if v is extreme, A_v must contain n linearly independent vectors.

Conversely, assume that A_v contains a subset $\bar{A}_v$ of n linearly independent vectors. Suppose that for some $y \in P$, $z \in P$, and $\alpha \in (0,1)$, we have $v = \alpha y + (1-\alpha)z$. Then, for all $a_j \in \bar{A}_v$,

$$b_j = a_j' v = \alpha a_j' y + (1-\alpha) a_j' z \le \alpha b_j + (1-\alpha) b_j = b_j.$$

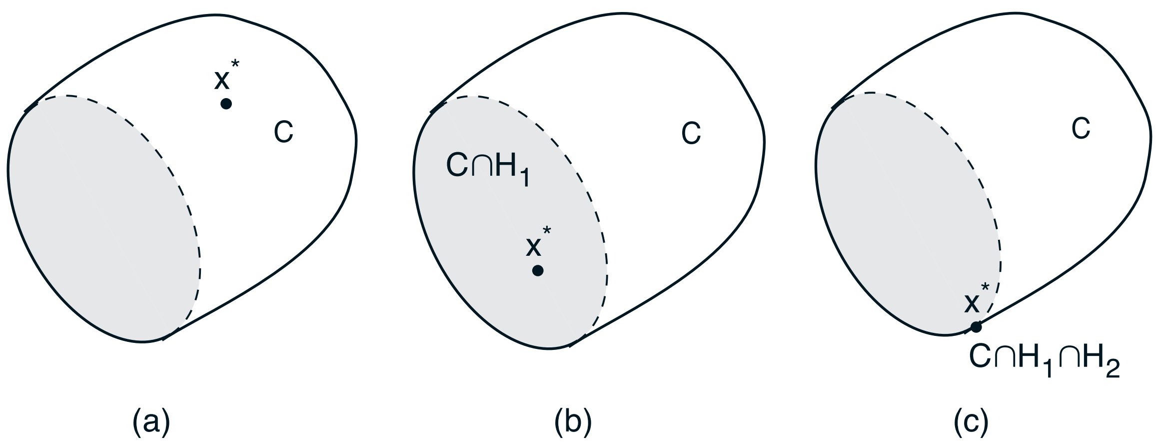
Thus, v, y, and z are all solutions of the system of n linearly independent equations

$$a_j' w = b_j, \qquad \forall\, a_j \in \bar{A}_v.$$

Hence, $v = y = z$, implying that v is an extreme point of P.

EXTREME POINTS AND CONCAVE MINIMIZATION

- Let C be a closed and convex set that has at least one extreme point. A concave function $f : C \mapsto \Re$ that attains a minimum over C attains the minimum at some extreme point of C.



Proof (abbreviated): If $x^* \in \mathrm{ri}(C)$ [see (a)], f must be constant over C, so it attains a minimum at an extreme point of C. If $x^* \notin \mathrm{ri}(C)$, there is a hyperplane H_1 that supports C and contains x^*.

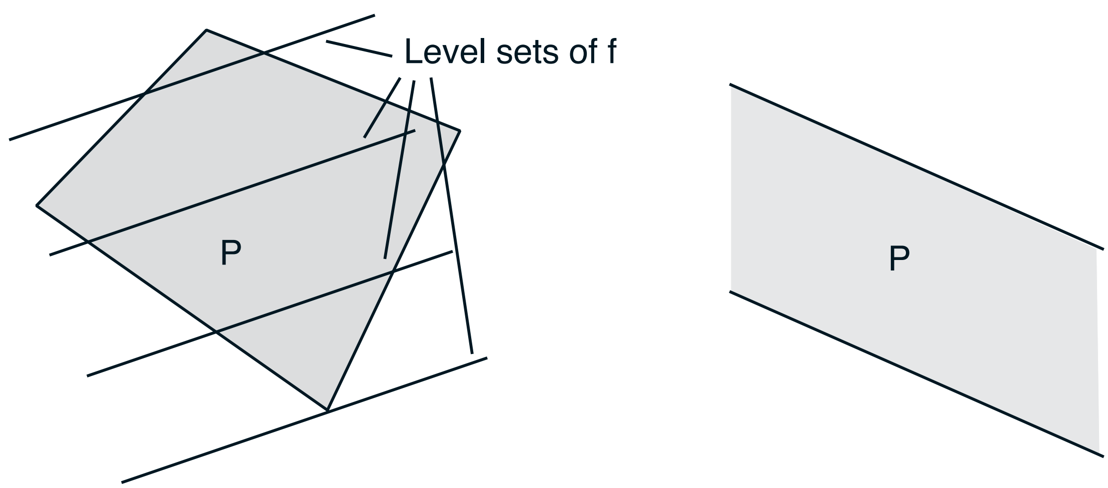
If $x^* \in \mathrm{ri}(C \cap H_1)$ [see (b)], then f must be constant over $C \cap H_1$, so it attains a minimum at an extreme point $C \cap H_1$. This optimal extreme point is also an extreme point of C. If $x^* \notin \mathrm{ri}(C \cap H_1)$, there is a hyperplane H_2 supporting $C \cap H_1$ through x^*. Continue until an optimal extreme point is obtained (which must also be an extreme point of C).

FUNDAMENTAL THEOREM OF LP

- Let P be a polyhedral set that has at least one extreme point. Then, if a linear function is bounded below over P, it attains a minimum at some extreme point of P.

Proof: Since the cost function is bounded below over P, it attains a minimum. The result now follows from the preceding theorem. **Q.E.D.**

- Two possible cases in LP: In (a) there is an extreme point; in (b) there is none.



(a) (b)

EXTREME POINTS AND INTEGER PROGRAMMING

- Consider a polyhedral set

$$P = \{x \mid Ax = b,\, c \le x \le d\},$$

where A is $m \times n$, $b \in \Re^m$, and $c, d \in \Re^n$. Assume that all components of A and b, c, and d are integer.

- Question: Under what conditions do the extreme points of P have integer components?

Definition: A square matrix with integer components is *unimodular* if its determinant is 0, 1, or -1. A rectangular matrix with integer components is *totally unimodular* if each of its square submatrices is unimodular.

Theorem: If A is totally unimodular, all the extreme points of P have integer components.

- Most important special case: Linear network optimization problems (with "single commodity" and no "side constraints"), where A is the, so-called, *arc incidence matrix* of a given directed graph.