LECTURE 26

LECTURE OUTLINE

- Additional Dual Methods
- Cutting Plane Methods
- Decomposition

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- Consider the primal problem

minimize $f(x)$
subject to $x \in X, \qquad g_j(x) \le 0, \qquad j = 1, \ldots, r,$

assuming $-\infty < f^* < \infty$.

- Dual problem: Maximize

$$q(\mu) = \inf_{x \in X} L(x, \mu) = \inf_{x \in X} \{f(x) + \mu' g(x)\}$$

subject to $\mu \in M = \{\mu \mid \mu \ge 0,\ q(\mu) > -\infty\}$.

CUTTING PLANE METHOD

- kth iteration, after μ^i and $g^i = g\big(x_{\mu^i}\big)$ have been generated for $i = 0, \ldots, k-1$: Solve

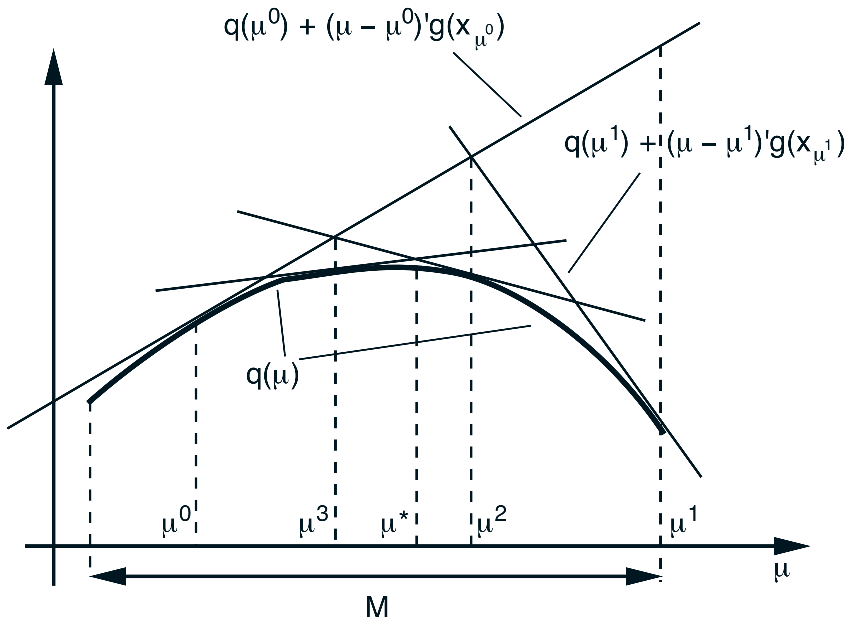
$$\max_{\mu \in M} Q^k(\mu)$$

where

$$Q^k(\mu) = \min_{i=0,\ldots,k-1} \big\{ q(\mu^i) + (\mu - \mu^i)' g^i \big\}.$$

Set

$$\mu^k = \arg\max_{\mu \in M} Q^k(\mu).$$

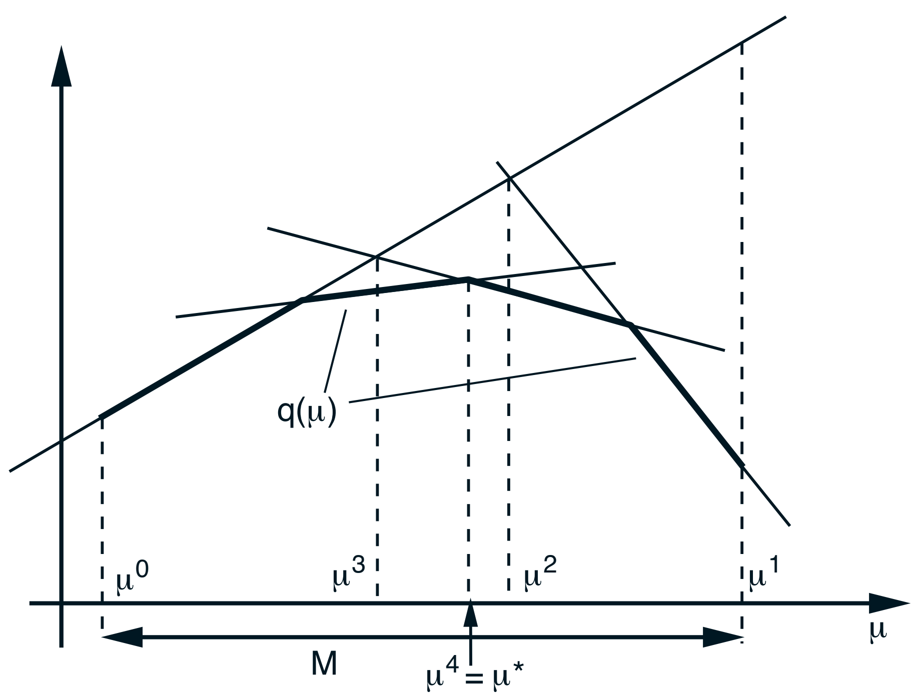


POLYHEDRAL CASE

$$q(\mu) = \min_{i \in I} \{ a_i' \mu + b_i \}$$

where I is a finite index set, and $a_i \in \Re^r$ and b_i are given.

- Then subgradient g^k in the cutting plane method is a vector a_{i^k} for which the minimum is attained.
- Finite termination expected.



CONVERGENCE

- Proposition: Assume that the min of Q_k over M is attained and that the sequence g^k is bounded. Then every limit point of a sequence $\{\mu^k\}$ generated by the cutting plane method is a dual optimal solution.

Proof: g^i is a subgradient of q at μ^i, so

$$q(\mu^i) + (\mu - \mu^i)' g^i \geq q(\mu), \qquad \forall\, \mu \in M,$$

$$Q^k(\mu^k) \geq Q^k(\mu) \geq q(\mu), \qquad \forall\, \mu \in M. \qquad (1)$$

- Suppose $\{\mu^k\}_K$ converges to $\bar{\mu}$. Then, $\bar{\mu} \in M$, and from (1), we obtain for all k and $i < k$,

$$q(\mu^i) + (\mu^k - \mu^i)' g^i \geq Q^k(\mu^k) \geq Q^k(\bar{\mu}) \geq q(\bar{\mu}).$$

- Take the limit as $i \to \infty$, $k \to \infty$, $i \in K$, $k \in K$,

$$\lim_{k \to \infty,\, k \in K} Q^k(\mu^k) = q(\bar{\mu}).$$

Combining with (1), $q(\bar{\mu}) = \max_{\mu \in M} q(\mu)$.

LAGRANGIAN RELAXATION

- Solving the dual of the separable problem

minimize $\sum_{j=1}^{J} f_j(x_j)$

subject to $x_j \in X_j, \;\; j = 1, \ldots, J, \quad \sum_{j=1}^{J} A_j x_j = b.$

- Dual function is

$$
\begin{aligned}
q(\lambda) &= \sum_{j=1}^{J} \min_{x_j \in X_j} \left\{ f_j(x_j) + \lambda' A_j x_j \right\} - \lambda' b \\
&= \sum_{j=1}^{J} \left\{ f_j\big(x_j(\lambda)\big) + \lambda' A_j x_j(\lambda) \right\} - \lambda' b
\end{aligned}
$$

where $x_j(\lambda)$ attains the min. A subgradient at λ is

$$
g_\lambda = \sum_{j=1}^{J} A_j x_j(\lambda) - b.
$$

DANTSIG-WOLFE DECOMPOSITION

- D-W decomposition method is just the cutting plane applied to the dual problem $\max_\lambda q(\lambda)$.
- At the kth iteration, we solve the "approximate dual"

$$\lambda^k = \arg\max_{\lambda \in \Re^r} Q^k(\lambda) \equiv \min_{i=0,\ldots,k-1} \left\{ q(\lambda^i) + (\lambda - \lambda^i)' g^i \right\}.$$

- Equivalent linear program in v and λ

maximize v

subject to $v \leq q(\lambda^i) + (\lambda - \lambda^i)' g^i, \quad i = 0, \ldots, k-1$

The dual of this (called *master problem*) is

$$\text{minimize} \quad \sum_{i=0}^{k-1} \xi^i \big(q(\lambda^i) - \lambda^{i\prime} g^i \big)$$

$$\text{subject to} \quad \sum_{i=0}^{k-1} \xi^i = 1, \qquad \sum_{i=0}^{k-1} \xi^i g^i = 0,$$

$$\xi^i \geq 0, \quad i = 0, \ldots, k-1,$$

DANTSIG-WOLFE DECOMPOSITION (CONT.)

- The master problem is written as

$$\text{minimize} \quad \sum_{j=1}^{J} \left(\sum_{i=0}^{k-1} \xi^i f_j\big(x_j(\lambda^i)\big) \right)$$

$$\text{subject to} \quad \sum_{i=0}^{k-1} \xi^i = 1, \qquad \sum_{j=1}^{J} A_j \left(\sum_{i=0}^{k-1} \xi^i x_j(\lambda^i) \right) = b,$$

$$\xi^i \geq 0, \quad i = 0, \ldots, k-1.$$

- The primal cost function terms $f_j(x_j)$ are approximated by

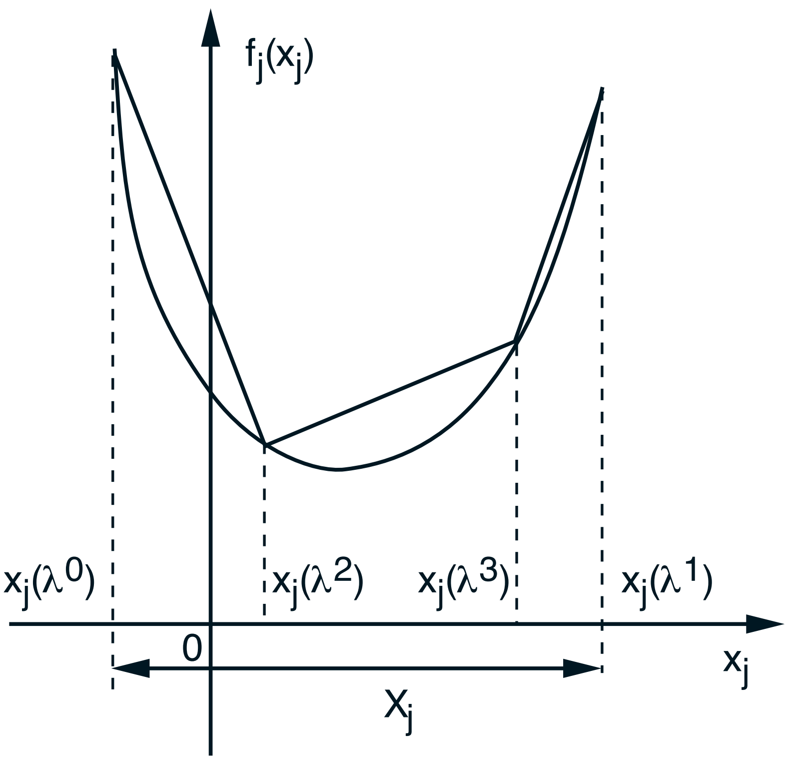
$$\sum_{i=0}^{k-1} \xi^i f_j\big(x_j(\lambda^i)\big)$$

- Vectors x_j are expressed as

$$\sum_{i=0}^{k-1} \xi^i x_j(\lambda^i)$$

GEOMETRICAL INTERPRETATION

- Geometric interpretation of the master problem (the dual of the approximate dual solved in the cutting plane method) is *inner linearization*.



- This is a "dual" operation to the one involved in the cutting plane approximation, which can be viewed as *outer linearization*.