LECTURE 18

LECTURE OUTLINE

- Convexity, geometric multipliers, and duality
- Relation of geometric and Lagrange multipliers
- The dual function and the dual problem
- Weak and strong duality
- Duality and geometric multipliers

GEOMETRICAL FRAMEWORK FOR MULTIPLIERS

- We start an alternative geometric approach to Lagrange multipliers and duality for the problem

$$\begin{array}{rl} \text{minimize} & f(x) \\ \text{subject to} & x \in X,\ g_1(x) \leq 0, \ldots, g_r(x) \leq 0 \end{array}$$

- We assume nothing on X, f, and g_j, except that

$$-\infty < f^* = \inf_{\substack{x \in X \\ g_j(x) \leq 0,\, j=1,\ldots,r}} f(x) < \infty$$

- A vector $\mu^* = (\mu_1^*, \ldots, \mu_r^*)$ is said to be a *geometric multiplier* if $\mu^* \geq 0$ and

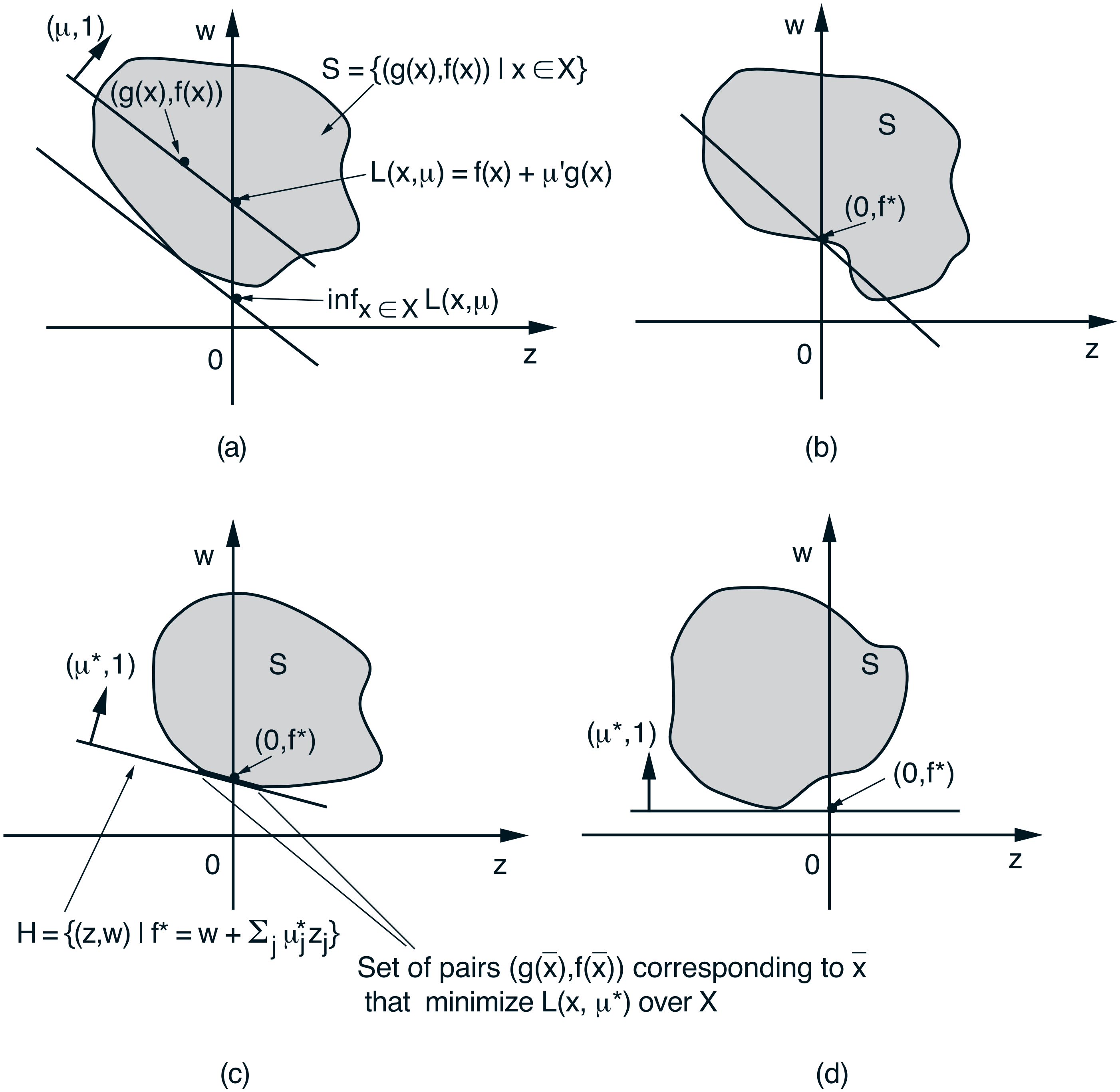
$$f^* = \inf_{x \in X} L(x, \mu^*),$$

where

$$L(x, \mu) = f(x) + \mu' g(x)$$

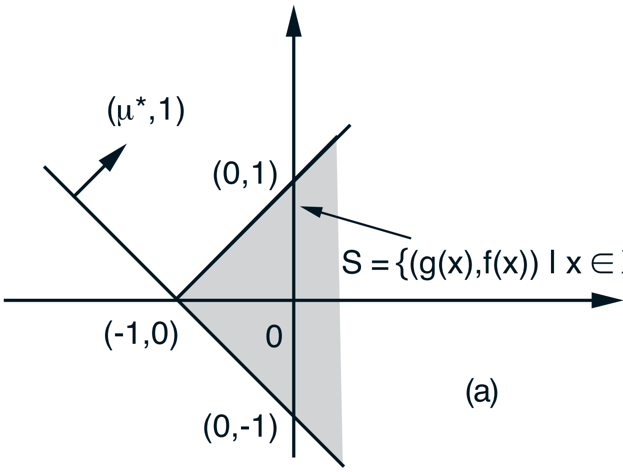
- Note that *a G-multiplier is associated with the problem and not with a specific local minimum.*

VISUALIZATION

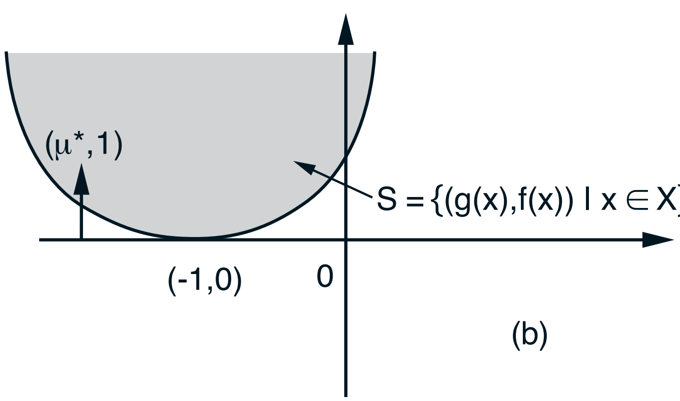


- Note: A G-multiplier solves a max-crossing problem whose min common problem has optimal value f^*.

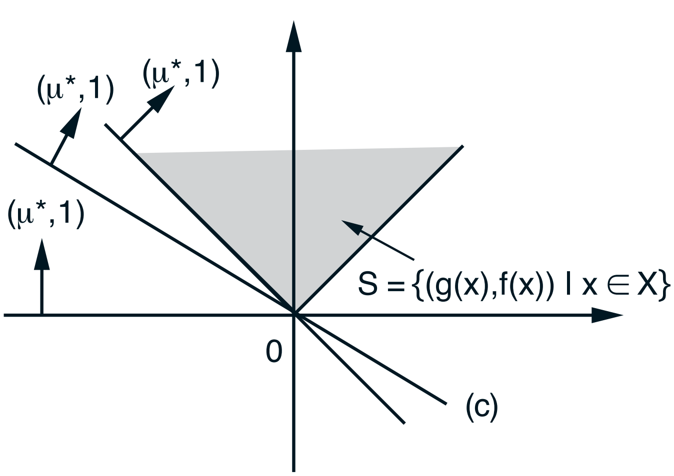
EXAMPLES: A G-MULTIPLIER EXISTS



$$\min\ f(x) = x_1 - x_2$$
$$\text{s.t. } g(x) = x_1 + x_2 - 1 \le 0$$
$$x \in X = \{(x_1,x_2) \mid x_1 \ge 0,\ x_2 \ge 0\}$$

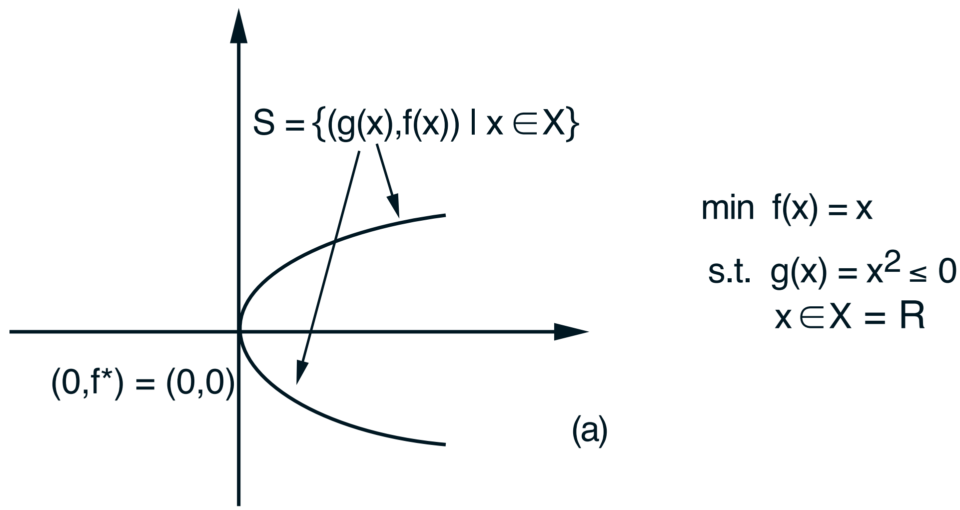


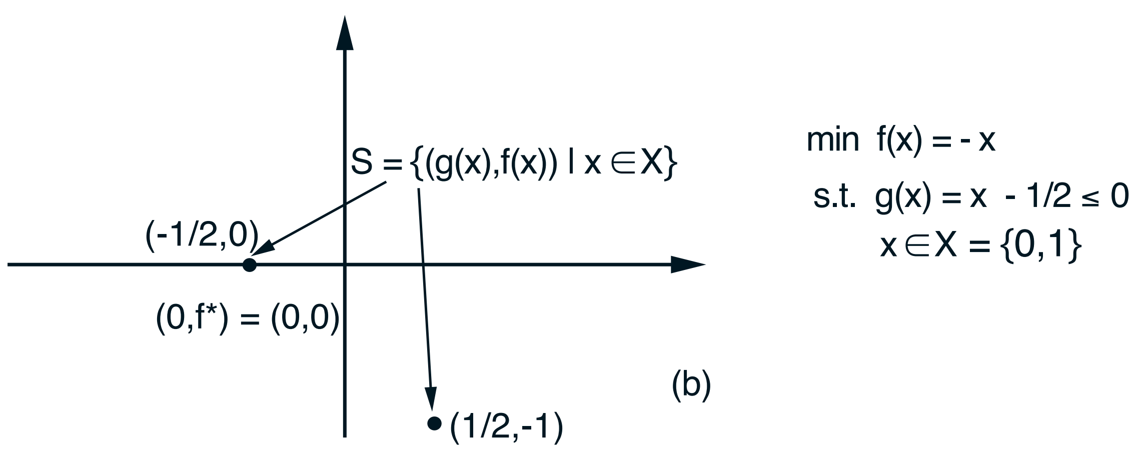
$$\min\ f(x) = (1/2)\,(x_1^2 + x_2^2)$$
$$\text{s.t. } g(x) = x_1 - 1 \le 0$$
$$x \in X = R^2$$



$$\min\ f(x) = |x_1| + x_2$$
$$\text{s.t. } g(x) = x_1 \le 0$$
$$x \in X = \{(x_1,x_2) \mid x_2 \ge 0\}$$

EXAMPLES: A G-MULTIPLIER DOESN'T EXIST





- **Proposition:** Let μ^* be a geometric multiplier. Then x^* is a global minimum of the primal problem if and only if x^* is feasible and

$$x^* = \arg\min_{x \in X} L(x, \mu^*), \quad \mu_j^* g_j(x^*) = 0, \quad j = 1, \ldots, r$$

RELATION BETWEEN G- AND L- MULTIPLIERS

- Assume the problem is convex (X closed and convex, and f and g_j are convex and differentiable over $\Re^n$), and x^* is a global minimum. Then the set of L-multipliers concides with the set of G-multipliers.

- For convex problems, the set of G-multipliers does not depend on the optimal solution x^* (it is the same for all x^*, and may be nonempty even if the problem has no optimal solution x^*).

- In general (for nonconvex problems):
  - Set of G-multipliers may be empty even if the set of L-multipliers is nonempty. [Example problem: $\min_{x=0}(-x^2)$]
  - "Typically" there is no G-multiplier if the set

    $$\big\{(u,w) \mid \text{for some } x \in X,\ g(x) \le u,\ f(x) \le w\big\}$$

    is nonconvex, which often happens if the problem is nonconvex.
  - The G-multiplier idea underlies duality even if the problem is nonconvex.

THE DUAL FUNCTION AND THE DUAL PROBLEM

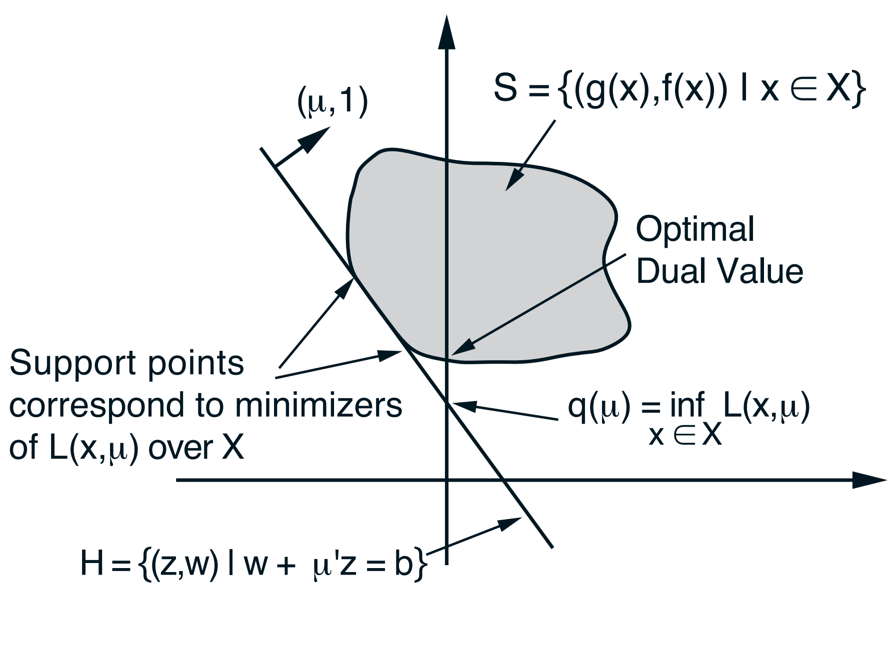
- The *dual problem* is

$$\begin{array}{ll} \text{maximize} & q(\mu) \\ \text{subject to} & \mu \geq 0, \end{array}$$

where q is the dual function

$$q(\mu) = \inf_{x \in X} L(x, \mu), \qquad \forall\, \mu \in \Re^r.$$

- Note: The dual problem is equivalent to a max-crossing problem.



WEAK DUALITY

- The $domain$ of q is

$$D_q = \{\mu \mid q(\mu) > -\infty\}.$$

- **Proposition:** q is concave, i.e., the domain D_q is a convex set and q is concave over D_q.

- **Proposition:** (Weak Duality Theorem) We have

$$q^* \leq f^*.$$

Proof: For all $\mu \geq 0$, and $x \in X$ with $g(x) \leq 0$, we have

$$q(\mu) = \inf_{z \in X} L(z, \mu) \leq f(x) + \sum_{j=1}^{r} \mu_j g_j(x) \leq f(x),$$

so

$$q^* = \sup_{\mu \geq 0} q(\mu) \leq \inf_{x \in X,\, g(x) \leq 0} f(x) = f^*.$$

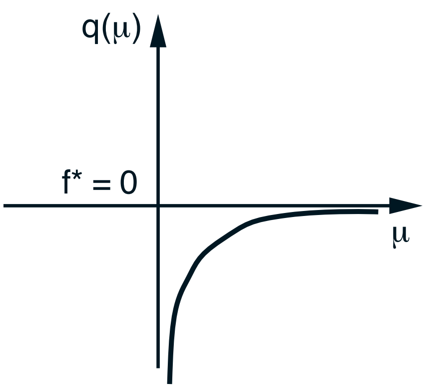
DUAL OPTIMAL SOLUTIONS AND G-MULTIPLIERS

Proposition: (a) If $q^* = f^*$, the set of G-multipliers is equal to the set of optimal dual solutions.
(b) If $q^* < f^*$, the set of G-multipliers is empty (so if there exists a G-multiplier, $q^* = f^*$).

Proof: By definition, $\mu^* \geq 0$ is a G-multiplier if $f^* = q(\mu^*)$. Since $q(\mu^*) \leq q^*$ and $q^* \leq f^*$,

$$\mu^* \geq 0 \text{ is a G-multiplier} \quad \text{iff} \quad q(\mu^*) = q^* = f^*$$

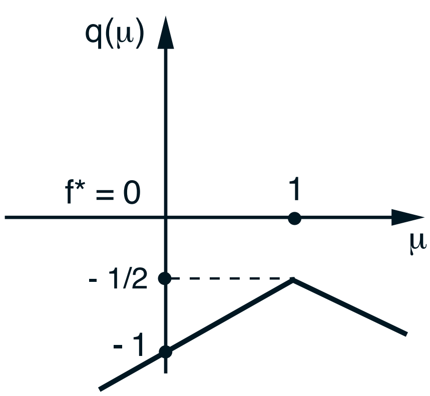
- Examples (dual functions for two problems with no G-multipliers, given earlier):



(a)

$$\min \ f(x) = x$$
$$\text{s.t.} \ g(x) = x^2 \leq 0$$
$$x \in X = R$$

$$q(\mu) = \min_{x \in R} \{x + \mu x^2\} = \begin{cases} -1/(4\mu) & \text{if } \mu > 0 \\ -\infty & \text{if } \mu \leq 0 \end{cases}$$



(b)

$$\min \ f(x) = -x$$
$$\text{s.t.} \ g(x) = x - 1/2 \leq 0$$
$$x \in X = \{0,1\}$$

$$q(\mu) = \min_{x \in \{0,1\}} \{-x + \mu(x - 1/2)\} = \min\{-\mu/2, \mu/2 - 1\}$$

DUALITY AND MINIMAX THEORY

- The primal and dual problems can be viewed in terms of minimax theory:

$$\text{Primal Problem} \;<=>\; \inf_{x\in X}\sup_{\mu\geq 0} L(x,\mu)$$

$$\text{Dual Problem} \;<=>\; \sup_{\mu\geq 0}\inf_{x\in X} L(x,\mu)$$

- Optimality Conditions: (x^*,μ^*) is an optimal solution/G-multiplier pair if and only if

$$\begin{aligned} x^*\in X,\;\; g(x^*)\leq 0, &\quad \text{(Primal Feasibility)},\\ \mu^*\geq 0, &\quad \text{(Dual Feasibility)},\\ x^*=\arg\min_{x\in X} L(x,\mu^*), &\quad \text{(Lagrangian Optimality)},\\ \mu_j^* g_j(x^*)=0,\quad j=1,\ldots,r, &\quad \text{(Compl. Slackness)}. \end{aligned}$$

- Saddle Point Theorem: (x^*,μ^*) is an optimal solution/G-multiplier pair if and only if $x^*\in X$, $\mu^*\geq 0$, and (x^*,μ^*) is a saddle point of the Lagrangian, in the sense that

$$L(x^*,\mu)\leq L(x^*,\mu^*)\leq L(x,\mu^*),\;\;\forall\, x\in X,\; \mu\geq 0$$

A CONVEX PROBLEM WITH A DUALITY GAP

- Consider the two-dimensional problem

$$\begin{aligned} &\text{minimize } f(x) \\ &\text{subject to } x_1 \le 0, \qquad x \in X = \{x \mid x \ge 0\}, \end{aligned}$$

where

$$f(x) = e^{-\sqrt{x_1 x_2}}, \qquad \forall\, x \in X,$$

and $f(x)$ is arbitrarily defined for $x \notin X$.

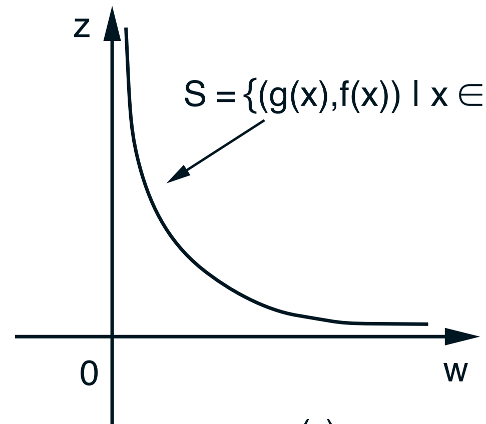
- f is convex over X (its Hessian is positive definite in the interior of X), and $f^* = 1$.

- Also, for all $\mu \ge 0$ we have

$$q(\mu) = \inf_{x \ge 0} \left\{ e^{-\sqrt{x_1 x_2}} + \mu x_1 \right\} = 0,$$

since the expression in braces is nonnegative for $x \ge 0$ and can approach zero by taking $x_1 \to 0$ and $x_1 x_2 \to \infty$. It follows that $q^* = 0$.

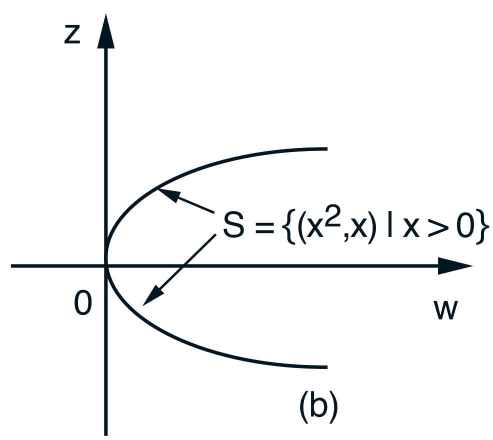
INFEASIBLE AND UNBOUNDED PROBLEMS



(a)

$$\min\ f(x) = 1/x$$
$$\text{s.t. } g(x) = x \le 0$$
$$x \in X = \{x \mid x > 0\}$$

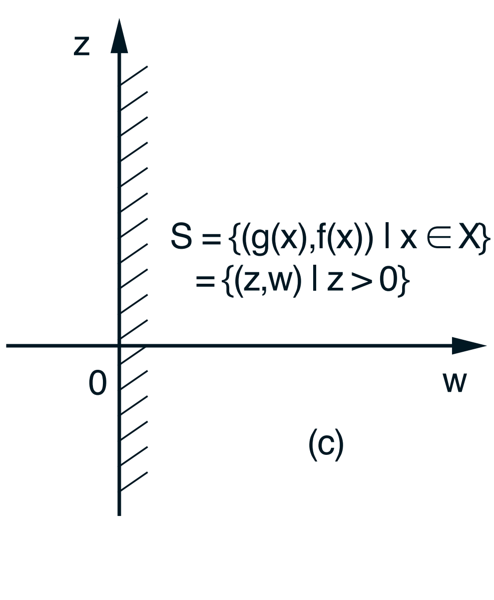
$$\boxed{f^* = \infty, \quad q^* = \infty}$$



(b)

$$\min\ f(x) = x$$
$$\text{s.t. } g(x) = x^2 \le 0$$
$$x \in X = \{x \mid x > 0\}$$

$$\boxed{f^* = \infty, \quad q^* = 0}$$



(c)

$$\min\ f(x) = x_1 + x_2$$
$$\text{s.t. } g(x) = x_1 \le 0$$
$$x \in X = \{(x_1, x_2) \mid x_1 > 0\}$$

$$\boxed{f^* = \infty, \quad q^* = -\infty}$$