LECTURE 21

LECTURE OUTLINE

- Fenchel Duality
- Conjugate Convex Functions
- Relation of Primal and Dual Functions
- Fenchel Duality Theorems

FENCHEL DUALITY FRAMEWORK

- Consider the problem

$$\begin{aligned}&\text{minimize} \;\; f_1(x) - f_2(x)\\ &\text{subject to} \;\; x \in X_1 \cap X_2,\end{aligned}$$

where f_1 and f_2 are real-valued functions on $\Re^n$, and X_1 and X_2 are subsets of $\Re^n$.

- Assume that $f^* < \infty$.
- Convert problem to

$$\begin{aligned}&\text{minimize} \;\; f_1(y) - f_2(z)\\ &\text{subject to} \;\; z = y, \qquad y \in X_1, \qquad z \in X_2,\end{aligned}$$

and dualize the constraint $z = y$:

$$\begin{aligned}q(\lambda) &= \inf_{y \in X_1,\, z \in X_2} \bigl\{ f_1(y) - f_2(z) + (z - y)'\lambda \bigr\}\\ &= \inf_{z \in X_2} \bigl\{ z'\lambda - f_2(z) \bigr\} - \sup_{y \in X_1} \bigl\{ y'\lambda - f_1(y) \bigr\}\\ &= g_2(\lambda) - g_1(\lambda)\end{aligned}$$

CONJUGATE FUNCTIONS

- The functions $g_1(\lambda)$ and $g_2(\lambda)$ are called the $conjugate\ convex$ and $conjugate\ concave$ functions corresponding to the pairs (f_1, X_1) and (f_2, X_2).

- An equivalent definition of g_1 is

$$g_1(\lambda) = \sup_{x \in \Re^n} \big\{x'\lambda - \tilde{f}_1(x)\big\},$$

where $\tilde{f}_1$ is the extended real-valed function

$$\tilde{f}_1(x) = \begin{cases} f_1(x) & \text{if } x \in X_1, \\ \infty & \text{if } x \notin X_1. \end{cases}$$

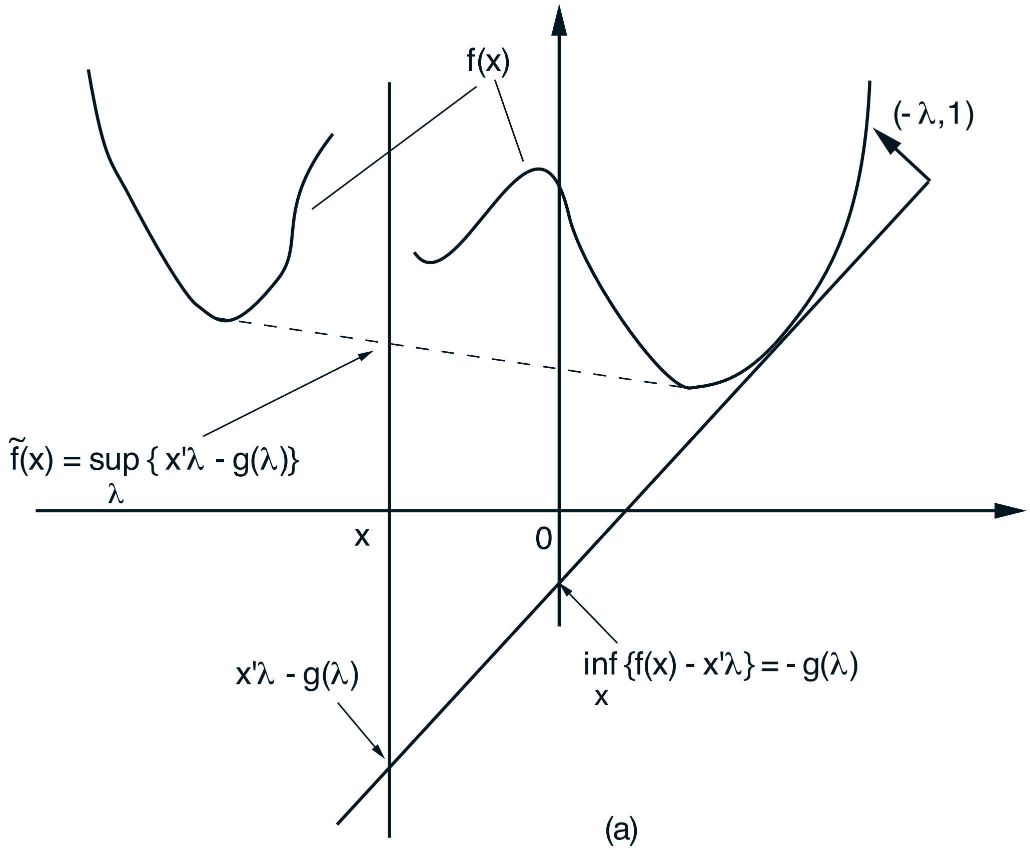
- We are led to consider the conjugate convex function of a general extended real-valued proper function $f : \Re^n \mapsto (-\infty, \infty]$:

$$g(\lambda) = \sup_{x \in \Re^n} \big\{x'\lambda - f(x)\big\}, \qquad \lambda \in \Re^n.$$

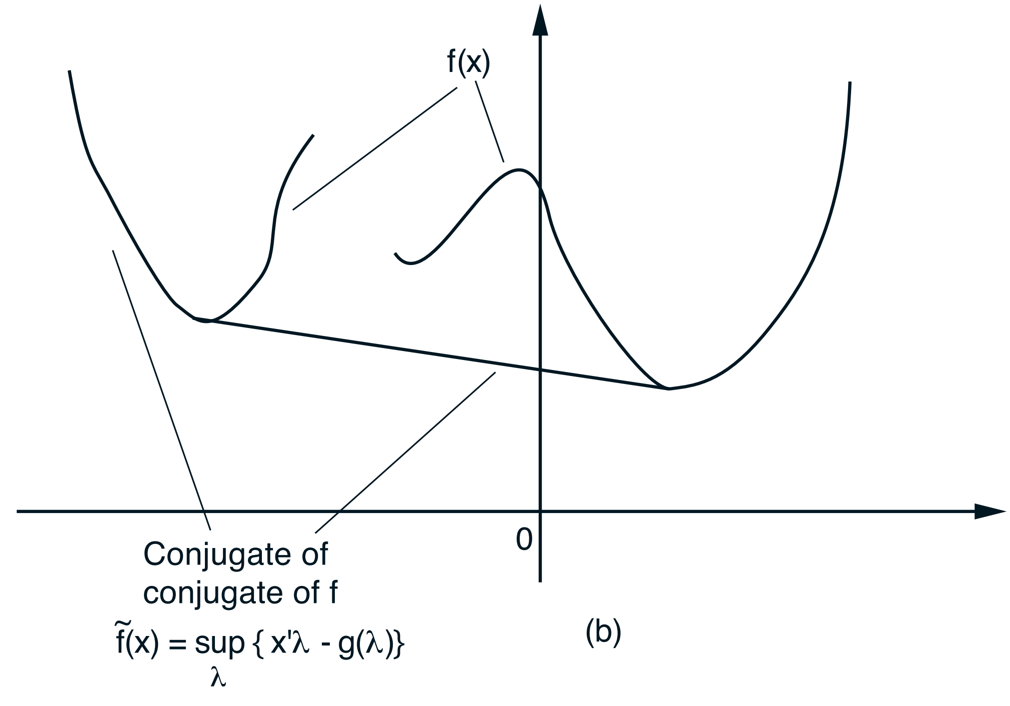
- Conjugate concave functions are defined through conjugate convex functions after appropriate sign reversals.

VISUALIZATION

$$g(\lambda) = \sup_{x \in \Re^n} \big\{ x'\lambda - f(x) \big\}, \qquad \lambda \in \Re^n$$



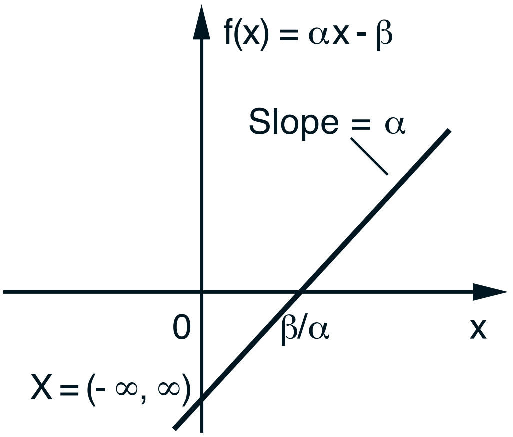
(a)

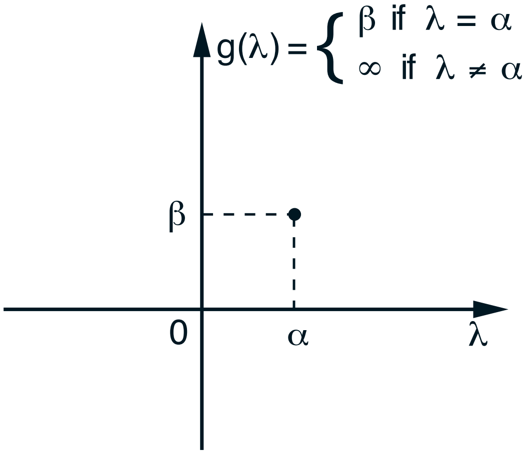


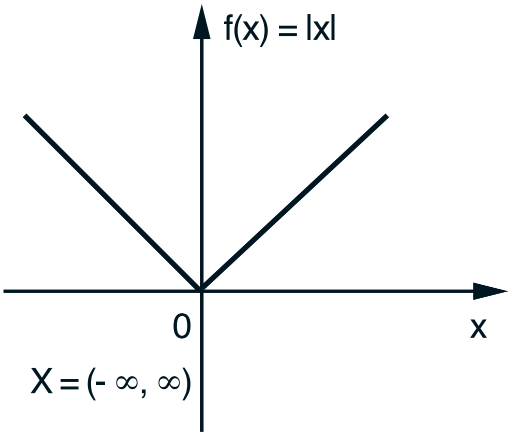
(b)

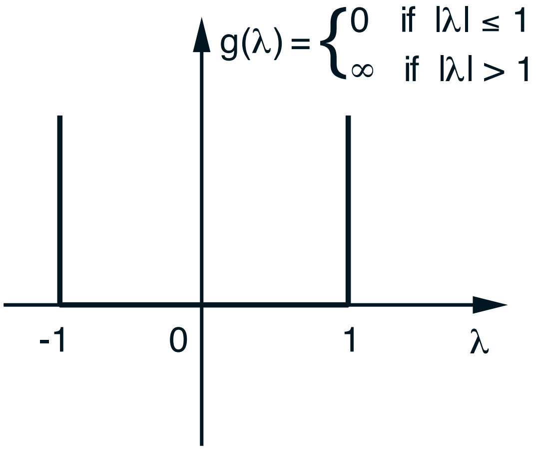
EXAMPLES OF CONJUGATE PAIRS

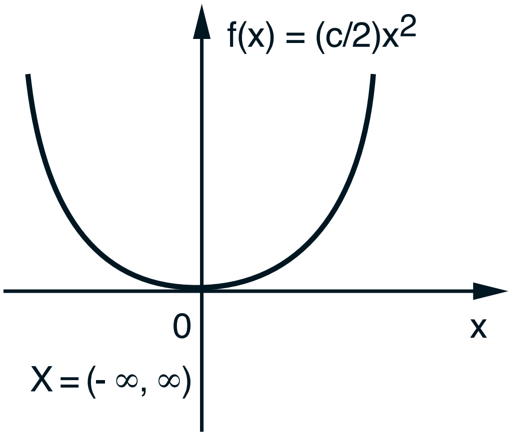
$$g(\lambda) = \sup_{x \in \Re^n} \big\{ x'\lambda - f(x) \big\}, \qquad \tilde{f}(x) = \sup_{\lambda \in \Re^n} \big\{ x'\lambda - g(\lambda) \big\}$$

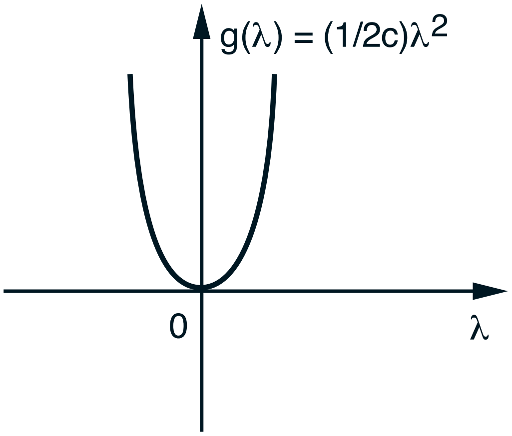












CONJUGATE OF THE CONJUGATE FUNCTION

- Two cases to consider:
  - f is a closed proper convex function.
  - f is a general extended real-valued proper function.

- We will see that for closed proper convex functions, the conjugacy operation is symmetric, i.e., *the congugate of* f *is a closed proper convex function, and the conjugate of the conjugate is* f*.*

- Leads to a symmetric/dual Fenchel duality theorem for the case where the functions involved are closed convex/concave.

- The result can be generalized:
  - The *convex closure of* f, is the function that has as epigraph the closure of the convex hull if epi(f) [also the smallest closed and convex set containing epi(f)].
  - The epigraph of the convex closure of f is the intersection of all closed halfspaces of $\Re^{n+1}$ that contain the epigraph of f.

CONJUGATE FUNCTION THEOREM

- Let $f : \Re^n \mapsto (-\infty, \infty]$ be a function, let $\hat{f}$ be its convex closure, let g be its convex conjugate, and consider the conjugate of g,

$$\tilde{f}(x) = \sup_{\lambda \in \Re^n} \big\{ \lambda' x - g(\lambda) \big\}, \qquad x \in \Re^n.$$

(a) We have

$$f(x) \geq \tilde{f}(x), \qquad \forall\, x \in \Re^n.$$

(b) If f is convex, then properness of any one of f, g, and $\tilde{f}$ implies properness of the other two.

(c) If f is closed proper and convex, then

$$f(x) = \tilde{f}(x), \qquad \forall\, x \in \Re^n.$$

(d) If $\hat{f}(x) > -\infty$ for all $x \in \Re^n$, then

$$\hat{f}(x) = \tilde{f}(x), \qquad \forall\, x \in \Re^n.$$

CONJUGACY OF PRIMAL AND DUAL FUNCTIONS

- Consider the problem

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in X, \qquad g_j(x) \leq 0, \quad j = 1, \ldots, r. \end{array}$$

- We showed in the previous lecture the following relation between primal and dual functions:

$$q(\mu) = \inf_{u \in \Re^r} \big\{ p(u) + \mu' u \big\}, \qquad \forall\, \mu \geq 0.$$

- Thus, $q(\mu) = -\sup_{u \in \Re^r} \big\{ -\mu' u - p(u) \big\}$ or

$$q(\mu) = -h(-\mu), \qquad \forall\, \mu \geq 0,$$

where h is the conjugate convex function of p:

$$h(\nu) = \sup_{u \in \Re^r} \big\{ \nu' u - p(u) \big\}.$$

INDICATOR AND SUPPORT FUNCTIONS

- The *indicator function* of a nonempty set is

$$\delta_X(x) = \begin{cases} 0 & \text{if } x \in X, \\ \infty & \text{if } x \notin X. \end{cases}$$

- The conjugate of δ_X, given by

$$\sigma_X(\lambda) = \sup_{x \in X} \lambda' x,$$

is called the *support function of* X.

- X has the same support function as $\mathrm{cl}\big(\mathrm{conv}(X)\big)$ (by the Conjugacy Theorem).

- If X is closed and convex, δ_X is closed and convex, and by the Conjugacy Theorem the conjugate of its support function is its indicator function.

- The support function satisfies

$$\sigma_X(\alpha\lambda) = \alpha\sigma_X(\lambda), \qquad \forall\ \alpha > 0,\ \forall\ \lambda \in \Re^n.$$

so its epigraph is a cone. Functions with this property are called *positively homogeneous*.

MORE ON SUPPORT FUNCTIONS

- For a cone C, we have

$$\sigma_C(\lambda) = \sup_{x \in C} \lambda' x = \begin{cases} 0 & \text{if } \lambda \in C^*, \\ \infty & \text{otherwise,} \end{cases}$$

i.e., the support function of a cone is the indicator function of its polar.

- The support function of a polyhedral set is a polyhedral function that is pos. homogeneous. The conjugate of a pos. homogeneous polyhedral function is the support function of some polyhedral set.

- A function can be equivalently specified in terms of its epigraph. As a consequence, we will see that the conjugate of a function can be specified in terms of the support function of its epigraph.

- The conjugate of f, can equivalently be written as $g(\lambda) = \sup_{(x,w)\in \mathrm{epi}(f)} \{x'\lambda - w\}$, so

$$g(\lambda) = \sigma_{\mathrm{epi}(f)}(\lambda, -1), \qquad \forall\, \lambda \in \Re^n.$$

- From this formula, we also obtain that the conjugate of a polyhedral function is polyhedral.