LECTURE 14

LECTURE OUTLINE

- Conical approximations
- Cone of feasible directions
- Tangent and normal cones
- Conditions for optimality

- A basic necessary condition:
  - If x^* minimizes a function $f(x)$ over $x \in X$, then for every $y \in \Re^n$, $\alpha^* = 0$ minimizes $g(\alpha) \equiv f(x + \alpha y)$ over the line subset

$$\{\alpha \mid x + \alpha y \in X\}.$$

- Special cases of this condition (f: differentiable):
  - $X = \Re^n$: $\nabla f(x^*) = 0$.
  - X is convex: $\nabla f(x^*)'(x - x^*) \geq 0$, $\forall\, x \in X$.
- We will aim for more general conditions.

CONE OF FEASIBLE DIRECTIONS

- Consider a subset X of $\Re^n$ and a vector $x \in X$.

- A vector $y \in \Re^n$ is a *feasible direction* of X at x if there exists an $\overline{\alpha} > 0$ such that $x + \alpha y \in X$ for all $\alpha \in [0, \overline{\alpha}]$.

- The set of all feasible directions of X at x is denoted by $F_X(x)$.

- $F_X(x)$ is a cone containing the origin. It need not be closed or convex.

- If X is convex, $F_X(x)$ consists of the vectors of the form $\alpha(\overline{x} - x)$ with $\alpha > 0$ and $\overline{x} \in X$.

- Easy optimality condition: If x^* minimizes a differentiable function $f(x)$ over $x \in X$, then

$$\nabla f(x^*)' y \geq 0, \qquad \forall\, y \in F_X(x^*).$$

- Difficulty: The condition may be vacuous because there may be no feasible directions (other than 0).

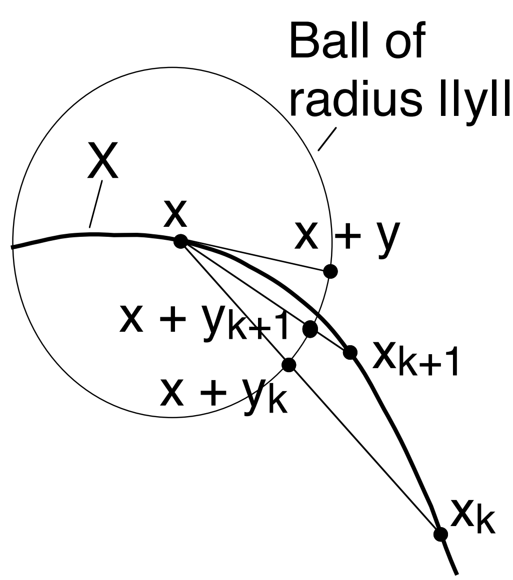
TANGENT CONE

- Consider a subset X of $\Re^n$ and a vector $x \in X$.

- A vector $y \in \Re^n$ is said to be a *tangent* of X at x if either $y = 0$ or there exists a sequence $\{x_k\} \subset X$ such that $x_k \neq x$ for all k and

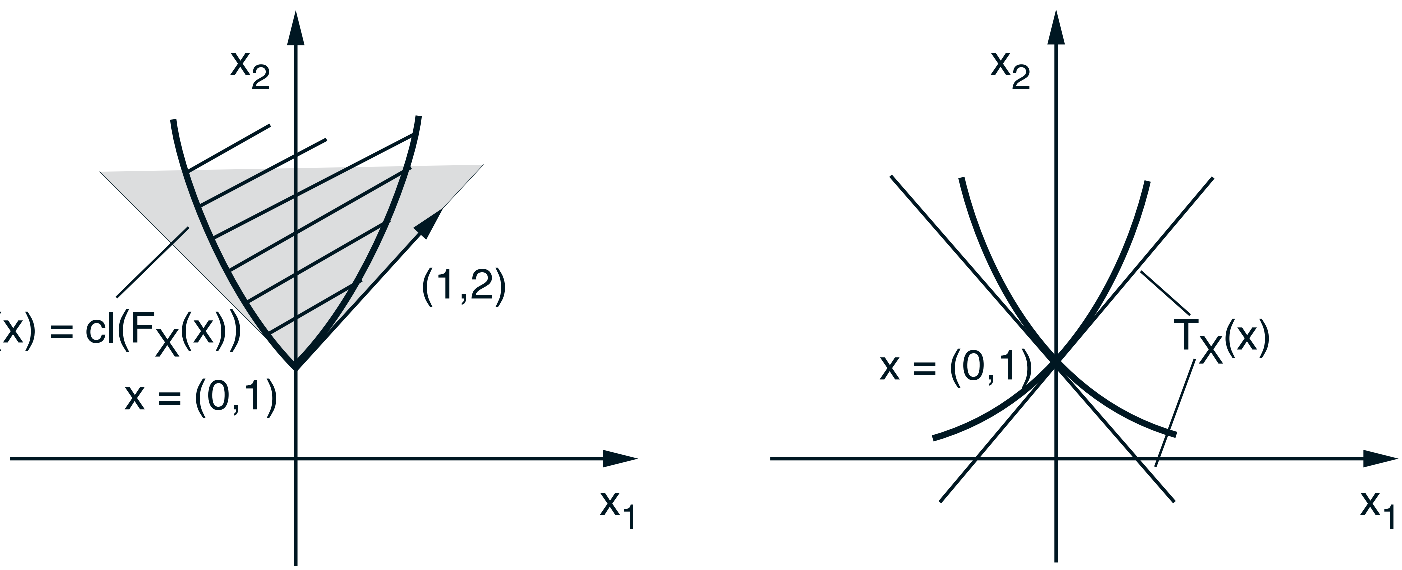
$$x_k \to x, \qquad \frac{x_k - x}{\|x_k - x\|} \to \frac{y}{\|y\|}.$$

- The set of all tangents of X at x is called the *tangent cone* of X at x, and is denoted by $T_X(x)$.



- y is a tangent of X at x iff there exists $\{x_k\} \subset X$ with $x_k \to x$, and a positive scalar sequence $\{\alpha_k\}$ such that $\alpha_k \to 0$ and $(x_k - x)/\alpha_k \to y$.

EXAMPLES



(a) (b)

- In (a), X is convex: The tangent cone $T_X(x)$ is equal to the closure of the cone of feas. directions $F_X(x)$.
- In (b), X is nonconvex: $T_X(x)$ is closed but not convex, while $F_X(x)$ consists of just the zero vector.
- In general, $F_X(x) \subset T_X(x)$.
- For X: polyhedral, $F_X(x) = T_X(x)$.

RELATION OF CONES

- Let X be a subset of $\Re^n$ and let x be a vector in X. The following hold.

  (a) $T_X(x)$ is a closed cone.

  (b) $\mathrm{cl}\big(F_X(x)\big) \subset T_X(x)$.

  (c) If X is convex, then $F_X(x)$ and $T_X(x)$ are convex, and we have

$$\mathrm{cl}\big(F_X(x)\big) = T_X(x).$$

Proof: (a) Let $\{y_k\}$ be a sequence in $T_X(x)$ that converges to some $y \in \Re^n$. We show that $y \in T_X(x)$ ...

(b) Every feasible direction is a tangent, so $F_X(x) \subset T_X(x)$. Since by part (a), $T_X(x)$ is closed, the result follows.

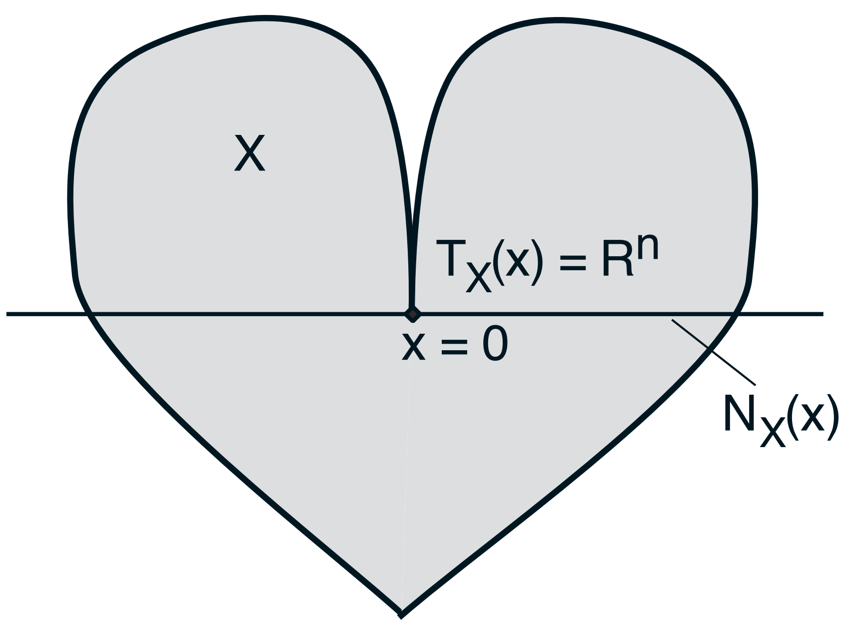
(c) Since X is convex, the set $F_X(x)$ consists of the vectors of the form $\alpha(\overline{x} - x)$ with $\alpha > 0$ and $\overline{x} \in X$. Verify definition of convexity ...

NORMAL CONE

- Consider subset X of $\Re^n$ and a vector $x \in X$.
- A vector $z \in \Re^n$ is said to be a *normal* of X at x if there exist sequences $\{x_k\} \subset X$ and $\{z_k\}$ with

$$x_k \to x, \qquad z_k \to z, \qquad z_k \in T_X(x_k)^*, \quad \forall\, k.$$

- The set of all normals of X at x is called the *normal cone* of X at x and is denoted by $N_X(x)$.
- Example:



- $N_X(x)$ is "usually equal" to the polar $T_X(x)^*$, but may differ at points of "discontinuity" of $T_X(x)$.

RELATION OF NORMAL AND POLAR CONES

- We have $T_X(x)^* \subset N_X(x)$.

- When $N_X(x) = T_X(x)^*$, we say that X is *regular* at x.

- If X is convex, then for all $x \in X$, we have

$$z \in T_X(x)^* \text{ if and only if } z'(\overline{x} - x) \leq 0, \quad \forall\, \overline{x} \in X.$$

Furthermore, X is regular at all $x \in X$. In particular, we have

$$T_X(x)^* = N_X(x), \qquad T_X(x) = N_X(x)^*.$$

- Note that convexity of $T_X(x)$ does not imply regularity if X at x.

- Important fact in nonsmooth analysis: If X is closed and regular at x, then

$$T_X(x) = N_X(x)^*.$$

In particular, $T_X(x)$ is convex.

OPTIMALITY CONDITIONS I

- Let $f : \Re^n \mapsto \Re$ be a smooth function. If x^* is a local minimum of f over a set $X \subset \Re^n$, then

$$\nabla f(x^*)' y \geq 0, \qquad \forall\, y \in T_X(x^*).$$

Proof: Let $y \in T_X(x^*)$ with $y \neq 0$. Then, there exist $\{\xi_k\} \subset \Re$ and $\{x_k\} \subset X$ such that $x_k \neq x^*$ for all k, $\xi_k \to 0$, $x_k \to x^*$, and

$$(x_k - x^*)/\|x_k - x^*\| = y/\|y\| + \xi_k.$$

By the Mean Value Theorem, we have for all k

$$f(x_k) = f(x^*) + \nabla f(\tilde{x}_k)'(x_k - x^*),$$

where $\tilde{x}_k$ is a vector that lies on the line segment joining x_k and x^*. Combining these equations,

$$f(x_k) = f(x^*) + (\|x_k - x^*\|/\|y\|)\nabla f(\tilde{x}_k)' y_k,$$

where $y_k = y + \|y\|\xi_k$. If $\nabla f(x^*)' y < 0$, since $\tilde{x}_k \to x^*$ and $y_k \to y$, for sufficiently large k, $\nabla f(\tilde{x}_k)' y_k < 0$ and $f(x_k) < f(x^*)$. This contradicts the local optimality of x^*.

OPTIMALITY CONDITIONS II

- Let $f : \Re^n \mapsto \Re$ be a convex function. A vector x^* minimizes f over a convex set X if and only if there exists a subgradient $d \in \partial f(x^*)$ such that

$$d'(x - x^*) \geq 0, \qquad \forall\, x \in X.$$

Proof: If for some $d \in \partial f(x^*)$ and all $x \in X$, we have $d'(x - x^*) \geq 0$, then, from the definition of a subgradient we have $f(x) - f(x^*) \geq d'(x - x^*)$ for all $x \in X$. Hence $f(x) - f(x^*) \geq 0$ for all $x \in X$.

Conversely, suppose that x^* minimizes f over X. Then, x^* minimizes f over the closure of X, and we have

$$f'(x^*; x - x^*) = \sup_{d \in \partial f(x^*)} d'(x - x^*) \geq 0, \ \forall\, x \in \mathrm{cl}(X).$$

Therefore,

$$\inf_{x \in \mathrm{cl}(X) \cap \{z \mid \|z - x^*\| \leq 1\}} \ \sup_{d \in \partial f(x^*)} d'(x - x^*) = 0.$$

Apply the saddle point theorem to conclude that ”infsup=supinf” and that the supremum is attained by some $d \in \partial f(x^*)$.

OPTIMALITY CONDITIONS III

• Let x^* be a local minimum of a function $f : \Re^n \mapsto \Re$ over a subset X of $\Re^n$. Assume that the tangent cone $T_X(x^*)$ is convex, and that f has the form

$$f(x) = f_1(x) + f_2(x),$$

where f_1 is convex and f_2 is smooth. Then

$$-\nabla f_2(x^*) \in \partial f_1(x^*) + T_X(x^*)^*.$$

• The convexity assumption on $T_X(x^*)$ (which is implied by regularity) is essential in general.

• Example: Consider the subset of $\Re^2$

$$X = \big\{(x_1, x_2) \mid x_1 x_2 = 0\big\}.$$

Then $T_X(0)^* = \{0\}$. Take f to be any convex nondifferentiable function for which $x^* = 0$ is a global minimum over X, but $x^* = 0$ is not an unconstrained global minimum. Such a function violates the necessary condition.