LECTURE 1
AN INTRODUCTION TO THE COURSE

LECTURE OUTLINE

- Convex and Nonconvex Optimization Problems
- Why is Convexity Important in Optimization
- Lagrange Multipliers and Duality
- Min Common/Max Crossing Duality

OPTIMIZATION PROBLEMS

$$\text{minimize } \; f(x)$$
$$\text{subject to } \; x \in C$$

Cost function $f : \Re^n \mapsto \Re$, constraint set C, e.g.,

$$C = X \cap \big\{ x \mid h_1(x) = 0, \ldots, h_m(x) = 0 \big\} \\ \cap \big\{ x \mid g_1(x) \leq 0, \ldots, g_r(x) \leq 0 \big\}$$

- Examples of problem classifications:
  - Continuous vs discrete
  - Linear vs nonlinear
  - Deterministic vs stochastic
  - Static vs dynamic

- Convex programming problems are those for which f is convex and C is convex (they are continuous problems)

- However, convexity permeates all of optimization, including discrete problems.

WHY IS CONVEXITY SO SPECIAL IN OPTIMIZATION

- A convex function has no local minima that are not global
- A convex set has a nonempty relative interior
- A convex set is connected and has feasible directions at any point
- A nonconvex function can be "convexified" while maintaining the optimality of its global minima
- The existence of a global minimum of a convex function over a convex set is conveniently characterized in terms of directions of recession
- A polyhedral convex set is characterized in terms of a finite set of extreme points and extreme directions
- A convex function is continuous and has nice differentiability properties
- Closed convex cones are self-dual with respect to polarity
- Convex, lower semicontinuous functions are self-dual with respect to conjugacy

CONVEXITY AND DUALITY

- A multiplier vector for the problem

minimize $f(x)$ subject to $g_1(x) \le 0, \ldots, g_r(x) \le 0$

is a $\mu^* = (\mu_1^*, \ldots, \mu_r^*) \ge 0$ such that

$$\inf_{g_j(x) \le 0,\, j=1,\ldots,r} f(x) = \inf_{x \in \Re^n} L(x, \mu^*)$$

where L is the Lagrangian function

$$L(x, \mu) = f(x) + \sum_{j=1}^{r} \mu_j g_j(x), \qquad x \in \Re^n,\ \mu \in \Re^r.$$

- Dual function (always concave)

$$q(\mu) = \inf_{x \in \Re^n} L(x, \mu)$$

- Dual problem: Maximize $q(\mu)$ over $\mu \ge 0$

KEY DUALITY RELATIONS

- Optimal primal value

$$f^* = \inf_{g_j(x) \le 0,\, j=1,\dots,r} f(x) = \inf_{x \in \Re^n} \sup_{\mu \ge 0} L(x, \mu)$$

- Optimal dual value

$$q^* = \sup_{\mu \ge 0} q(\mu) = \sup_{\mu \ge 0} \inf_{x \in \Re^n} L(x, \mu)$$

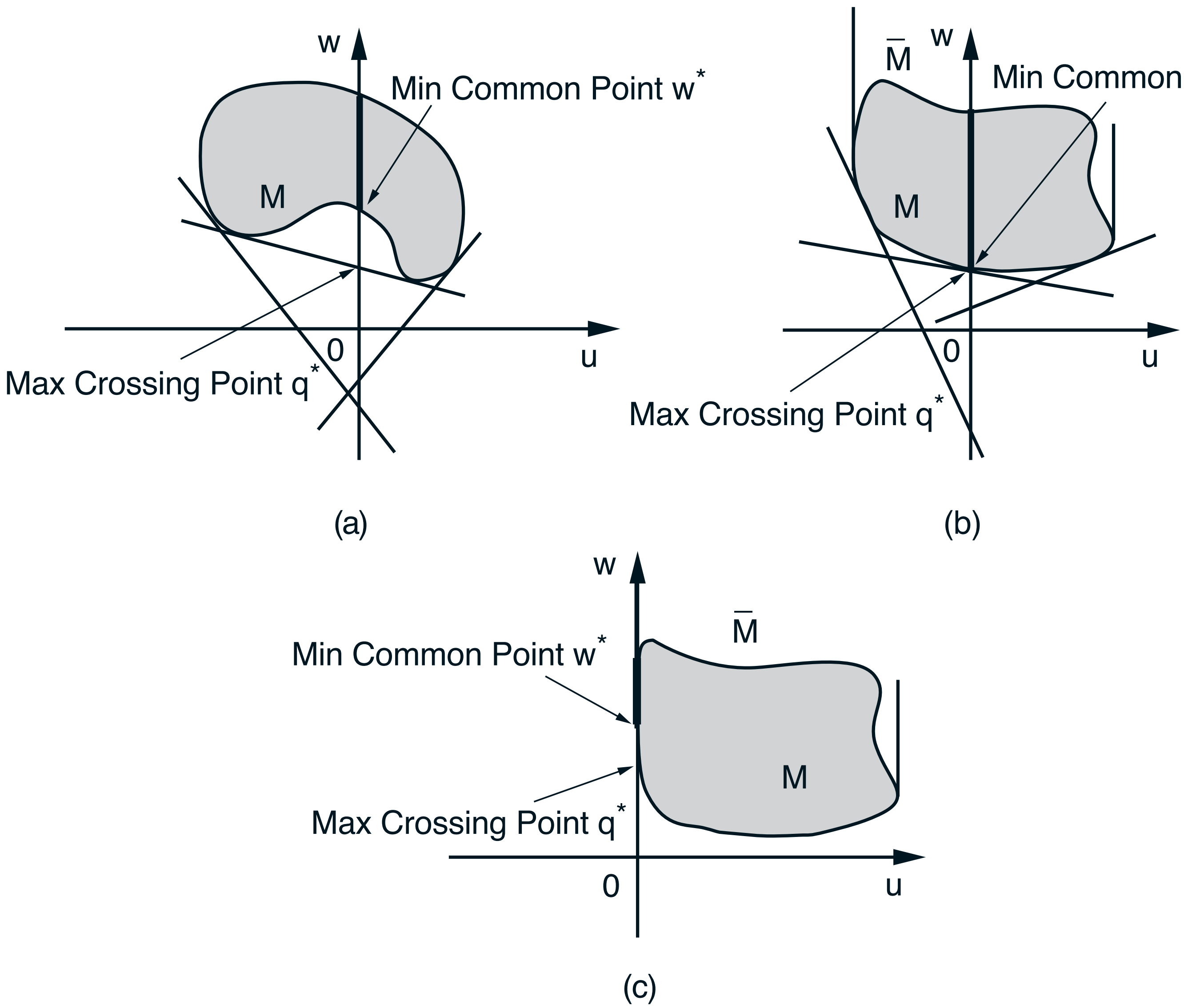
- We always have $q^* \le f^*$ (weak duality - important in discrete optimization problems)

- Under favorable circumstances (convexity in the primal problem):
  - We have $q^* = f^*$
  - Optimal solutions of the dual problem are multipliers for the primal problem

- This opens a wealth of analytical and computational possibilities, and insightful interpretations

- Note that the equality of "$\sup \inf$" and "$\inf \sup$" is a key issue in minimax theory and game theory.

MIN COMMON/MAX CROSSING DUALITY



- All of duality theory and all of (convex/concave) minimax theory can be developed/explained in terms of this one figure
- The machinery of convex analysis is needed to flesh out this figure

COURSE OUTLINE

1) Basic Convexity Concepts (4): Convex and affine hulls. Closure, relative interior, and continuity. Recession cones.
2) Convexity and Optimization (4): Directions of recession and existence of optimal solutions. Hyperplanes. Min common/max crossing duality. Saddle points and minimax theory.
3) Polyhedral Convexity (3): Polyhedral sets. Extreme points. Polyhedral aspects of optimization. Polyhedral aspects of duality.
4) Subgradients (3): Subgradients. Conical approximations. Optimality conditions.
5) Lagrange Multipliers (3): Fritz John theory. Pseudonormality and constraint qualifications.
6) Lagrangian Duality (3): Constrained optimization duality. Linear and quadratic programming duality. Duality theorems.
7) Conjugate Duality (2): Conjugacy and the Fenchel duality theorem. Exact penalties.
8) Dual Computational Methods (3): Classical subgradient and cutting plane methods. Application in Lagrangian relaxation and combinatorial optimization.

WHAT TO EXPECT FROM THIS COURSE

- Requirements: Homework and a term paper
- We aim:
  - To develop insight and deep understanding of a fundamental optimization topic
  - To treat rigorously an important branch of applied math, and to provide some appreciation of the research in the field
- Mathematical level:
  - Prerequisites are linear algebra (preferably abstract) and real analysis (a course in each)
  - Proofs will matter ... but the rich geometry of the subject helps guide the mathematics
- Applications:
  - They are many and pervasive ... but don't expect much in this course. The book by Boyd and Vandenberghe describes a lot of practical convex optimization models (see http://www.stanford.edu/ boyd/cvxbook.html)
  - You can do your term paper on an application area

A NOTE ON THESE SLIDES

- These slides are a teaching aid, not a text
- Don't expect a rigorous mathematical development
- The statements of theorems are fairly precise, but the proofs are not
- Many proofs have been omitted or greatly abbreviated
- Figures are meant to convey and enhance ideas, not to express them precisely
- The omitted proofs and a much fuller discussion can be found in the "Convex Analysis" text