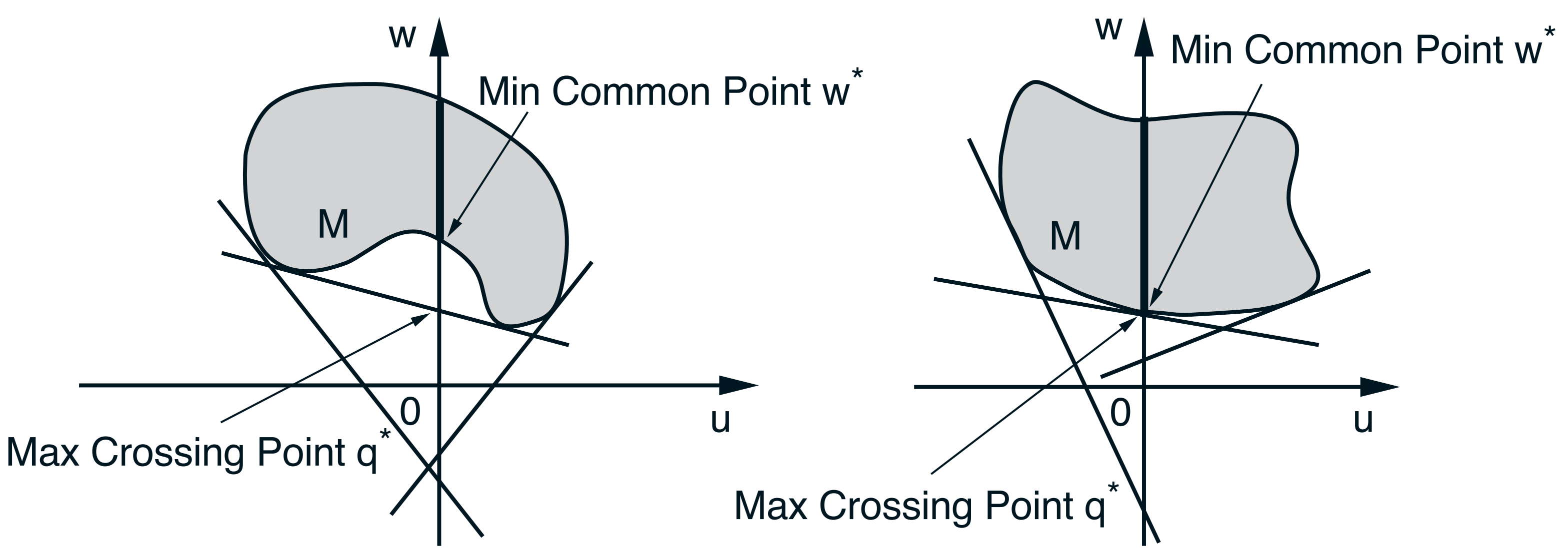
LECTURE 8

LECTURE OUTLINE

- Min Common / Max Crossing problems
- Weak duality
- Strong duality
- Existence of optimal solutions
- Minimax problems



WEAK DUALITY

- Optimal value of the min common problem:

$$w^* = \inf_{(0,w)\in M} w.$$

- Max crossing problem: Focus on hyperplanes with normals $(\mu, 1)$ whose crossing point ξ satisfies

$$\xi \le w + \mu' u, \qquad \forall\, (u,w) \in M.$$

Max crossing problem is to maximize ξ subject to $\xi \le \inf_{(u,w)\in M}\{w + \mu' u\}$, $\mu \in \Re^n$, or

$$\text{maximize} \;\; q(\mu) \stackrel{\triangle}{=} \inf_{(u,w)\in M}\{w + \mu' u\}$$

$$\text{subject to} \;\; \mu \in \Re^n.$$

- For all $(u,w) \in M$ and $\mu \in \Re^n$,

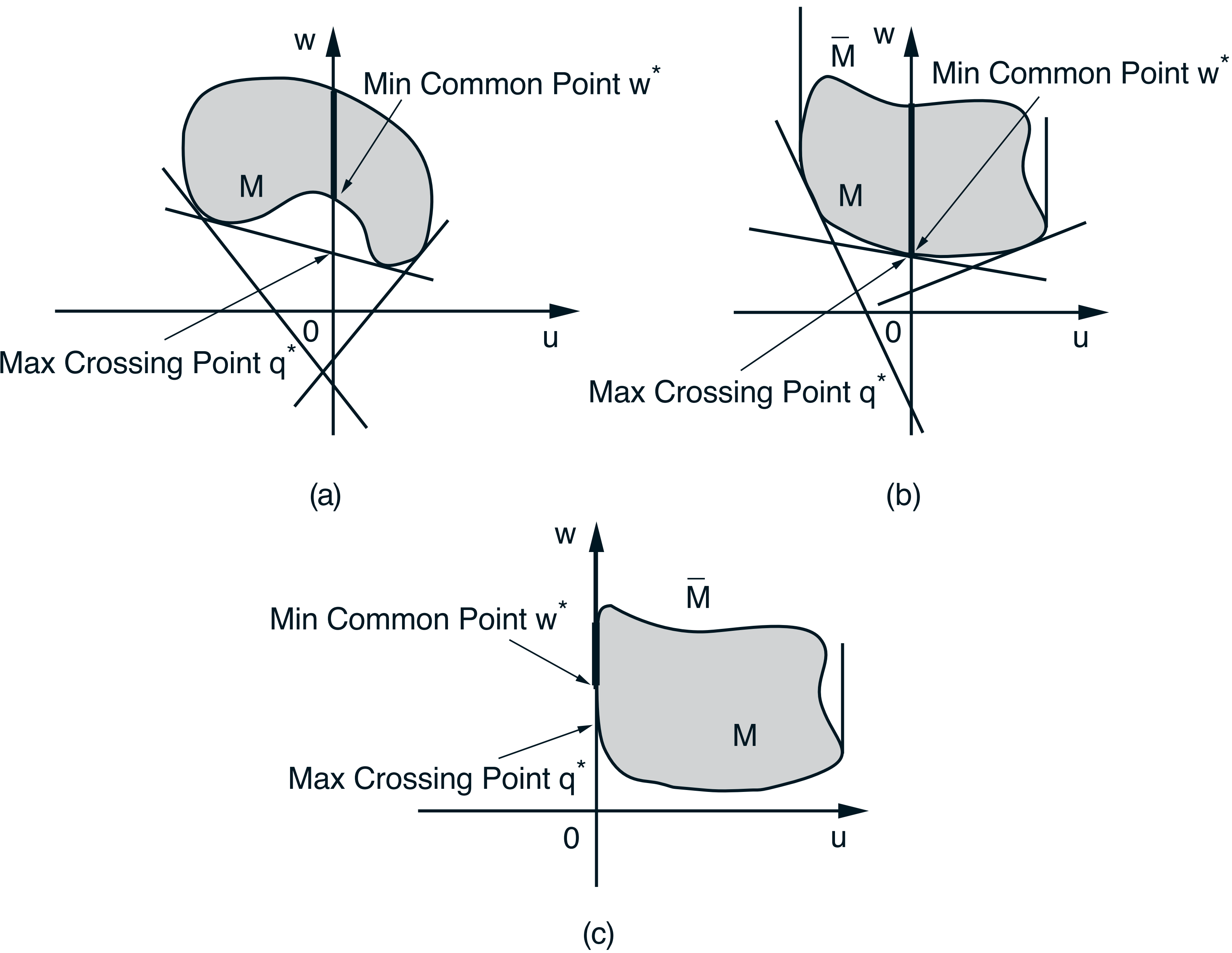
$$q(\mu) = \inf_{(u,w)\in M}\{w + \mu' u\} \le \inf_{(0,w)\in M} w = w^*,$$

so by taking the supremum of the left-hand side over $\mu \in \Re^n$, we obtain $q^* \le w^*$.

- Note that q is concave and upper-semicontinuous.

STRONG DUALITY

- Question: Under what conditions do we have $q^* = w^*$ and the supremum in the max crossing problem is attained?



DUALITY THEOREMS

- Assume that $w^* < \infty$ and that the set

$$\overline{M} = \Big\{(u,w) \mid \text{there exists } \overline{w} \text{ with } \overline{w} \le w \text{ and } (u,\overline{w}) \in M\Big\}$$

is convex.

- *Min Common/Max Crossing Theorem I*: We have $q^* = w^*$ if and only if for every sequence $\big\{(u_k, w_k)\big\} \subset M$ with $u_k \to 0$, there holds $w^* \le \liminf_{k\to\infty} w_k$.

- *Min Common/Max Crossing Theorem II*: Assume in addition that $-\infty < w^*$ and that the set

$$D = \big\{u \mid \text{there exists } w \in \Re \text{ with } (u,w) \in \overline{M}\big\}$$

contains the origin in its relative interior. Then $q^* = w^*$ and there exists a vector $\mu \in \Re^n$ such that $q(\mu) = q^*$. If D contains the origin in its interior, the set of all $\mu \in \Re^n$ such that $q(\mu) = q^*$ is compact.

- *Min Common/Max Crossing Theorem III*: Involves polyhedral assumptions, and will be developed later.

PROOF OF THEOREM I

- Assume that for every sequence $\{(u_k, w_k)\} \subset M$ with $u_k \to 0$, there holds $w^* \leq \liminf_{k\to\infty} w_k$. If $w^* = -\infty$, then $q^* = -\infty$, by weak duality, so assume that $-\infty < w^*$. Steps of the proof:
  (1) $\overline{M}$ does not contain any vertical lines.
  (2) $(0, w^* - \epsilon) \notin \mathrm{cl}(\overline{M})$ for any $\epsilon > 0$.
  (3) There exists a nonvertical hyperplane strictly separating $(0, w^* - \epsilon)$ and $\overline{M}$. This hyperplane crosses the $(n+1)$st axis at a vector $(0, \xi)$ with $w^* - \epsilon \leq \xi \leq w^*$, so $w^* - \epsilon \leq q^* \leq w^*$. Since ϵ can be arbitrarily small, it follows that $q^* = w^*$.

Conversely, assume that $q^* = w^*$. Let $\{(u_k, w_k)\} \subset M$ be such that $u_k \to 0$. Then,

$$q(\mu) = \inf_{(u,w)\in M} \{w + \mu' u\} \leq w_k + \mu' u_k, \quad \forall\, k,\ \forall\, \mu \in \Re^n.$$

Taking the limit as $k \to \infty$, we obtain $q(\mu) \leq \liminf_{k\to\infty} w_k$, for all $\mu \in \Re^n$, implying that

$$w^* = q^* = \sup_{\mu\in\Re^n} q(\mu) \leq \liminf_{k\to\infty} w_k.$$

PROOF OF THEOREM II

- Note that $(0, w^*)$ is not a relative interior point of $\overline{M}$. Therefore, by the Proper Separation Theorem, there exists a hyperplane that passes through $(0, w^*)$, contains $\overline{M}$ in one of its closed halfspaces, but does not fully contain $\overline{M}$, i.e., there exists (μ, β) such that

$$\beta w^* \leq \mu' u + \beta w, \qquad \forall\, (u, w) \in \overline{M},$$

$$\beta w^* < \sup_{(u,w)\in\overline{M}} \{\mu' u + \beta w\}.$$

Since for any $(\overline{u}, \overline{w}) \in M$, the set $\overline{M}$ contains the halfline $\big\{(\overline{u}, w) \mid \overline{w} \leq w\big\}$, it follows that $\beta \geq 0$. If $\beta = 0$, then $0 \leq \mu' u$ for all $u \in D$. Since $0 \in \mathrm{ri}(D)$ by assumption, we must have $\mu' u = 0$ for all $u \in D$ a contradiction. Therefore, $\beta > 0$, and we can assume that $\beta = 1$. It follows that

$$w^* \leq \inf_{(u,w)\in\overline{M}} \{\mu' u + w\} = q(\mu) \leq q^*.$$

Since the inequality $q^* \leq w^*$ holds always, we must have $q(\mu) = q^* = w^*$.

MINIMAX PROBLEMS

Given $\phi : X \times Z \mapsto \Re$, where $X \subset \Re^n$, $Z \subset \Re^m$ consider

$$\begin{aligned} &\text{minimize} \ \sup_{z \in Z} \phi(x, z) \\ &\text{subject to} \ x \in X \end{aligned}$$

and

$$\begin{aligned} &\text{maximize} \ \inf_{x \in X} \phi(x, z) \\ &\text{subject to} \ z \in Z. \end{aligned}$$

- Some important contexts:
  - Worst-case design. Special case: Minimize over $x \in X$

$$\max\{f_1(x), \ldots, f_m(x)\}$$

  - Duality theory and zero sum game theory (see next two slides)

- We will study minimax problems using the min common/max crossing framework

MAIN DUALITY STRUCTURE

- For the problem

minimize $f(x)$
subject to $x \in X, \qquad g_j(x) \leq 0, \quad j = 1, \ldots, r$

introduce the Lagrangian function

$$L(x, \mu) = f(x) + \sum_{j=1}^{r} \mu_j g_j(x)$$

- Primal problem (equivalent to the original)

$$\min_{x \in X} \sup_{\mu \geq 0} L(x, \mu) = \begin{cases} f(x) & \text{if } g(x) \leq 0, \\ \infty & \text{otherwise,} \end{cases}$$

- Dual problem

$$\max_{\mu \geq 0} \inf_{x \in X} L(x, \mu)$$

- Key duality question: Is it true that

$$\sup_{\mu \geq 0} \inf_{x \in \Re^n} L(x, \mu) = \inf_{x \in \Re^n} \sup_{\mu \geq 0} L(x, \mu)$$

ZERO SUM GAMES

- Two players: 1st chooses $i \in \{1, \ldots, n\}$, 2nd chooses $j \in \{1, \ldots, m\}$.

- If moves i and j are selected, the 1st player gives a_{ij} to the 2nd.

- Mixed strategies are allowed: The two players select probability distributions

$$x = (x_1, \ldots, x_n), \qquad z = (z_1, \ldots, z_m)$$

over their possible moves.

- Probability of (i, j) is $x_i z_j$, so the expected amount to be paid by the 1st player

$$x' A z = \sum_{i,j} a_{ij} x_i z_j$$

where A is the $n \times m$ matrix with elements a_{ij}.

- Each player optimizes his choice against the worst possible selection by the other player. So
  - 1st player minimizes $\max_z x' A z$
  - 2nd player maximizes $\min_x x' A z$

MINIMAX INEQUALITY

- We always have

$$\sup_{z \in Z} \inf_{x \in X} \phi(x, z) \leq \inf_{x \in X} \sup_{z \in Z} \phi(x, z)$$

[for every $\overline{z} \in Z$, write

$$\inf_{x \in X} \phi(x, \overline{z}) \leq \inf_{x \in X} \sup_{z \in Z} \phi(x, z)$$

and take the sup over $\overline{z} \in Z$ of the left-hand side].

- This is called the *minimax inequality*. When it holds as an equation, it is called the *minimax equality*.

- The minimax equality need not hold in general.

- When the minimax equality holds, it often leads to interesting interpretations and algorithms.

- The minimax inequality is often the basis for interesting bounding procedures.