LECTURE 5

LECTURE OUTLINE

- Global and local minima
- Weierstrass' theorem
- The projection theorem
- Recession cones of convex functions
- Existence of optimal solutions

WEIERSTRASS' THEOREM

- Let $f : \Re^n \mapsto (-\infty, \infty]$ be a closed proper function. Assume one of the following:

  (1) $\mathrm{dom}(f)$ is compact.

  (2) f is coercive.

  (3) There exists a scalar γ such that the level set
  $$\{x \mid f(x) \leq \gamma\}$$
  is nonempty and compact.

Then X^*, the set of minima of f over $\Re^n$, is nonempty and compact.

Proof: Let $f^* = \inf_{x \in \Re^n} f(x)$, and let $\{\gamma_k\}$ be a scalar sequence such that $\gamma_k \downarrow f^*$. We have

$$X^* = \cap_{k=0}^{\infty} \{x \mid f(x) \leq \gamma_k\}.$$

Under each of the assumptions, each set $\{x \mid f(x) \leq \gamma_k\}$ is nonempty and compact, so X^* is the intersection of a nested sequence of nonempty and compact sets. Hence, X^* is nonempty and compact. **Q.E.D.**

SPECIAL CASES

- Most common application of Weierstrass' Theorem:

Minimize a function $f : \Re^n \mapsto \Re$ over a set X.

Just apply the theorem to

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in X, \\ \infty & \text{otherwise.} \end{cases}$$

- **Result:** The set of minima of f over X is nonempty and compact if X is closed, f is lower semicontinuous over X (implying that $\tilde{f}$ is closed), and one of the following conditions holds:

  (1) X is bounded.

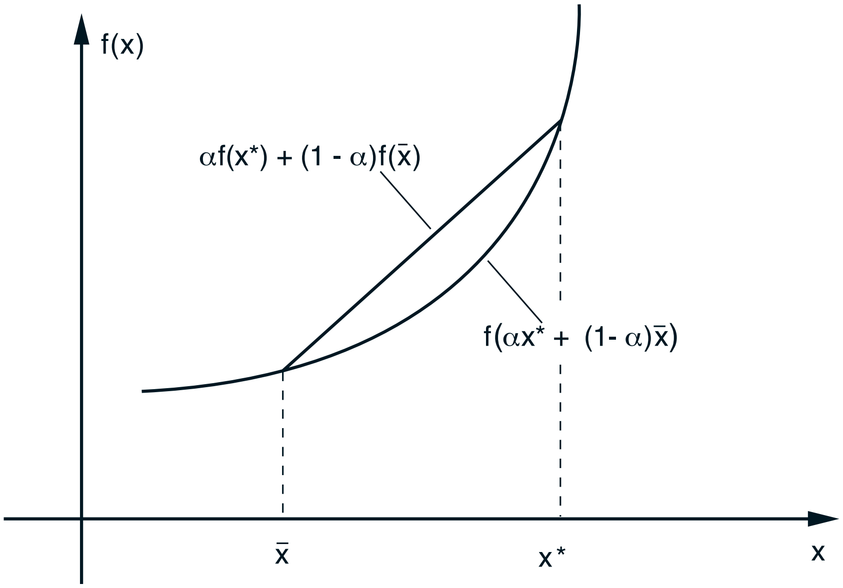
  (2) f is coercive.

  (3) Some level set $\{x \in X \mid f(x) \leq \gamma\}$ is nonempty and compact.

LOCAL AND GLOBAL MINIMA

- $x^* \in X$ is a *local minimum of* f *over* X if there exists some $\epsilon > 0$ such that $f(x^*) \le f(x)$ for all $x \in X$ with $\|x - x^*\| \le \epsilon$.
- If X is a convex subset of $\Re^n$ and $f : X \mapsto (-\infty, \infty]$ is a proper convex function, then a local minimum of f over X is also a global minimum. If in addition f is strictly convex, then there exists at most one global minimum of f.

Proof:



Assume that x^* is a local minimum that is not global. Choose $\overline{x} \in X$ with $f(\overline{x}) < f(x^*)$. By convexity, for all $\alpha \in (0,1)$,

$$f\big(\alpha x^* + (1-\alpha)\overline{x}\big) \le \alpha f(x^*) + (1-\alpha) f(\overline{x}) < f(x^*).$$

PROJECTION THEOREM

- Let C be a nonempty closed convex set in $\Re^n$.

  (a) For every $x \in \Re^n$, there exists a unique vector $P_C(x)$ that minimizes $\|z - x\|$ over all $z \in C$ (called the *projection of x on C*).

  (b) For every $x \in \Re^n$, a vector $z \in C$ is equal to $P_C(x)$ if and only if

$$(y - z)'(x - z) \leq 0, \qquad \forall\, y \in C.$$

  In the case where C is an affine set, the above condition is equivalent to

$$x - z \in S^\perp,$$

  where S is the subspace that is parallel to C.

  (c) The function $f : \Re^n \mapsto C$ defined by $f(x) = P_C(x)$ is continuous and nonexpansive, i.e.,

$$\big\|P_C(x) - P_C(y)\big\| \leq \|x - y\|, \qquad \forall\, x, y \in \Re^n.$$

PROOF OF THE PROJECTION THEOREM

- (a) Fix x and let w be some element of C. Minimizing $\|x-z\|$ over all $z \in C$ is equivalent to minimizing the continuous function $g(z) = \|z-x\|^2$ over the set of all $z \in C$ such that $\|x-z\| \leq \|x-w\|$, which is a compact set. Hence there exists a minimizing vector by Weierstrass, which is unique since $\|\cdot\|^2$ is a strictly convex function.

(b) For all y and z in C, we have

$$\begin{aligned}\|y-x\|^2 &= \|y-z\|^2 + \|z-x\|^2 - 2(y-z)'(x-z)\\ &\geq \|z-x\|^2 - 2(y-z)'(x-z).\end{aligned}$$

Therefore, if $(y-z)'(x-z) \leq 0$ for all $y \in C$, then $\|y-x\|^2 \geq \|z-x\|^2$ for all $y \in C$, implying that $z = P_C(x)$.

Conversely, let $z = P_C(x)$, consider any $y \in C$, and for $\alpha > 0$, let $y_\alpha = \alpha y + (1-\alpha)z$. We have

$$0 \leq \frac{\partial}{\partial \alpha}\left\{\|x-y_\alpha\|^2\right\}\Big|_{\alpha=0} = \cdots = -2(y-z)'(x-z).$$

RECESSION CONE OF LEVEL SETS

- *Proposition*: Let $f : \Re^n \mapsto (-\infty, \infty]$ be a closed proper convex function and consider the level sets $V_\gamma = \big\{x \mid f(x) \leq \gamma\big\}$, where γ is a scalar. Then:

  (a) All the nonempty level sets V_γ have the same recession cone, given by

  $$R_{V_\gamma} = \big\{y \mid (y, 0) \in R_{\mathrm{epi}(f)}\big\}.$$

  (b) If one nonempty level set V_γ is compact, then all nonempty level sets are compact.

Proof: For all γ for which V_γ is nonempty,

$$\big\{(x, \gamma) \mid x \in V_\gamma\big\} = \mathrm{epi}(f) \cap \big\{(x, \gamma) \mid x \in \Re^n\big\}.$$

The recession cone of the set on the left is $\big\{(y, 0) \mid y \in R_{V_\gamma}\big\}$. The recession cone of the set on the right is the intersection of $R_{\mathrm{epi}}(f)$ and the recession cone of $\big\{(x, \gamma) \mid x \in \Re^n\big\}$. Thus we have

$$\big\{(y, 0) \mid y \in R_{V_\gamma}\big\} = \big\{(y, 0) \mid (y, 0) \in R_{\mathrm{epi}(f)}\big\},$$

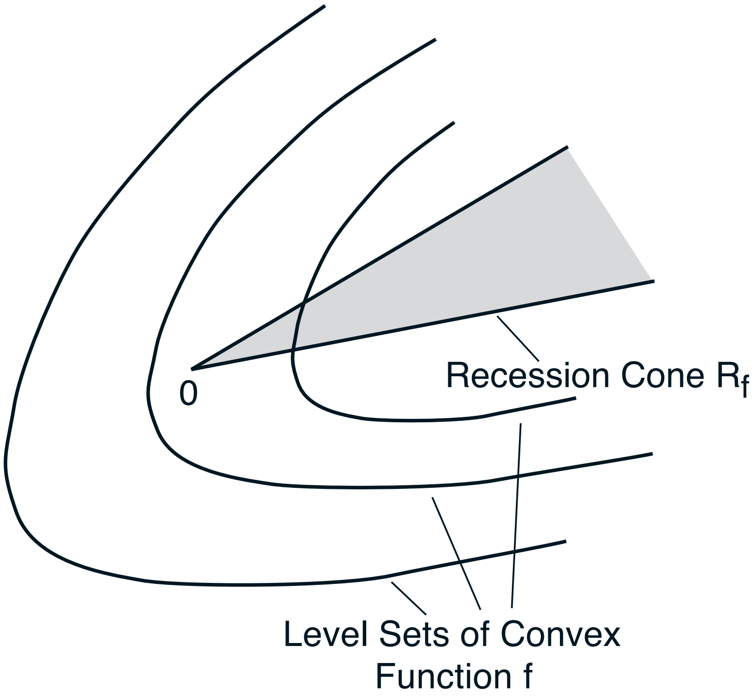
from which the result follows.

RECESSION CONE OF A CONVEX FUNCTION

- For a closed proper convex function $f : \Re^n \mapsto (-\infty, \infty]$, the (common) recession cone of the nonempty level sets

$$V_\gamma = \big\{x \mid f(x) \leq \gamma\big\}, \qquad a \in \Re,$$

is called the *recession cone of* f, and is denoted by R_f. Each $y \in R_f$ is called a *direction of recession of* f.



DESCENT BEHAVIOR OF A CONVEX FUNCTION

EXISTENCE OF SOLUTIONS - BOUNDED CASE

- Let X be a nonempty closed convex subset of $\Re^n$, and let $f : \Re^n \mapsto (-\infty, \infty]$ be a closed proper convex function such that $X \cap \mathrm{dom}(f) \neq \emptyset$. Then X^*, the set of minima of f over X, is nonempty and compact if and only if X and f have no common nonzero direction of recession.

Proof: Recall: A nonempty closed convex set is compact iff it has no direction of recession.

Let $f^* = \inf_{x \in \Re^n} f(x)$, let $\{\gamma_k\}$ be a scalar sequence such that $\gamma_k \downarrow f^*$, and note that

$$X^* = \cap_{k=0}^{\infty} \big(X \cap \big\{ x \mid f(x) \leq \gamma_k \big\} \big).$$

If X and f have no common nonzero direction of recession, the sets $X \cap \big\{ x \mid f(x) \leq \gamma_k \big\}$ are nonempty and compact, so X^* is nonempty and compact.

Conversely, if X^* is nonempty and compact, since $X^* = X \cap \big\{ x \mid f(x) \leq f^* \big\}$, it follows that X and $\big\{ x \mid f(x) \leq f^* \big\}$ have no common direction of recession, so X and f have no common direction of recession. **Q.E.D.**

EXISTENCE OF SOLUTIONS - UNBOUNDED CASE

- To address the case where the optimal solution set may be unbounded (e.g., linear and quadratic programming), we need set intersection theorems that do not rely on compactness.

- Given a family of nonempty closed sets $\{C_k\}$ with $C_{k+1} \subset C_k$ for all k, when is $\cap_{k=0}^{\infty} C_k \neq \emptyset$?

- The text shows (Section 1.5) that this is so if the C_k are convex and their recession cones/lineality spaces satisfy $\mathbf{one}$ of the following:
  - $\cap_{k=0}^{\infty} R_{C_k} = \cap_{k=0}^{\infty} L_{C_k}$
  - $C_k = X \cap \overline{C}_k$, where $\overline{C}_k$ is closed convex, X is of the form $X = \{x \mid a_j' x \leq b_j,\, j = 1, \ldots, r\}$, and the recession cones/lineality spaces of the $\overline{C}_k$ satisfy

    $$R_X \cap \left(\cap_{k=0}^{\infty} R_{\overline{C}_k} \right) \subset \cap_{k=0}^{\infty} L_{\overline{C}_k}$$

  - $C_k = X \cap \{x \mid x'Qx + a'x + b \leq w_k\}$, where Q is pos. semidefinite, a is a vector, b is a scalar, $w_k \downarrow 0$, and X is specified by a finite number of convex quadratic inequalities.