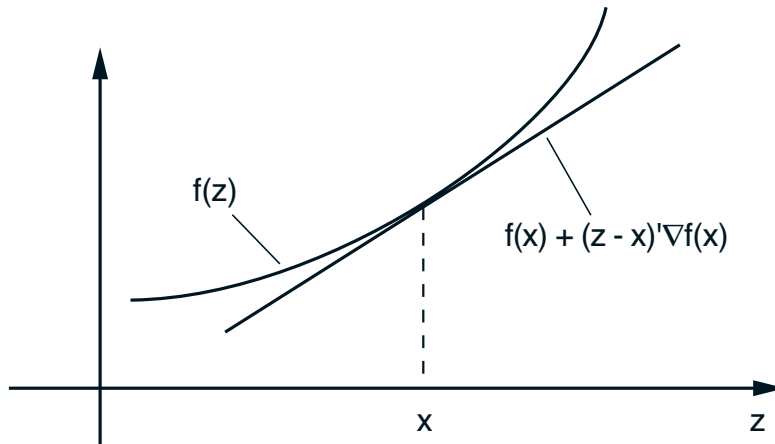


LECTURE 3

LECTURE OUTLINE

- Differentiable Convex Functions
- Convex and Affine Hulls
- Caratheodory's Theorem
- Closure, Relative Interior, Continuity

DIFFERENTIABLE CONVEX FUNCTIONS



• Let $C \subset \mathbb{R}^n$ be a convex set and let $f : \mathbb{R}^n \mapsto \mathbb{R}$ be differentiable over \mathbb{R}^n .

(a) The function f is convex over C if and only if

$$f(z) \geq f(x) + (z - x)' \nabla f(x), \quad \forall x, z \in C.$$

(b) If the inequality is strict whenever $x \neq z$, then f is strictly convex over C , i.e., for all $\alpha \in (0, 1)$ and $x, y \in C$, with $x \neq y$

$$f(\alpha x + (1 - \alpha)y) < \alpha f(x) + (1 - \alpha)f(y)$$

TWICE DIFFERENTIABLE CONVEX FUNCTIONS

- Let C be a convex subset of \mathbb{R}^n and let $f : \mathbb{R}^n \mapsto \mathbb{R}$ be twice continuously differentiable over \mathbb{R}^n .
 - (a) If $\nabla^2 f(x)$ is positive semidefinite for all $x \in C$, then f is convex over C .
 - (b) If $\nabla^2 f(x)$ is positive definite for all $x \in C$, then f is strictly convex over C .
 - (c) If C is open and f is convex over C , then $\nabla^2 f(x)$ is positive semidefinite for all $x \in C$.

Proof: (a) By mean value theorem, for $x, y \in C$

$$f(y) = f(x) + (y-x)' \nabla f(x) + \frac{1}{2} (y-x)' \nabla^2 f(x + \alpha(y-x)) (y-x)$$

for some $\alpha \in [0, 1]$. Using the positive semidefiniteness of $\nabla^2 f$, we obtain

$$f(y) \geq f(x) + (y-x)' \nabla f(x), \quad \forall x, y \in C.$$

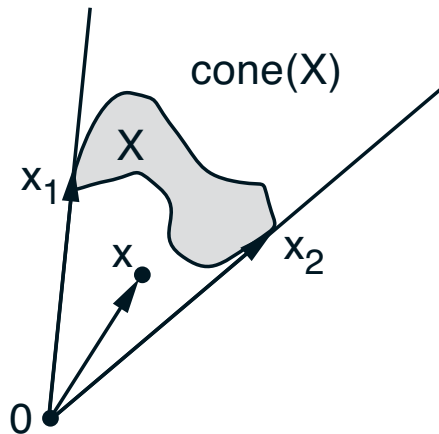
From the preceding result, f is convex.

(b) Similar to (a), we have $f(y) > f(x) + (y-x)' \nabla f(x)$ for all $x, y \in C$ with $x \neq y$, and we use the preceding result.

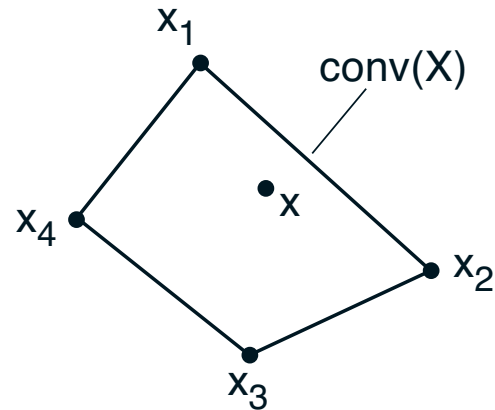
CONVEX AND AFFINE HULLS

- Given a set $X \subset \mathbb{R}^n$:
- A *convex combination* of elements of X is a vector of the form $\sum_{i=1}^m \alpha_i x_i$, where $x_i \in X$, $\alpha_i \geq 0$, and $\sum_{i=1}^m \alpha_i = 1$.
- The *convex hull* of X , denoted $\text{conv}(X)$, is the intersection of all convex sets containing X (also the set of all convex combinations from X).
- The *affine hull* of X , denoted $\text{aff}(X)$, is the intersection of all affine sets containing X (an affine set is a set of the form $\bar{x} + S$, where S is a subspace). Note that $\text{aff}(X)$ is itself an affine set.
- A *nonnegative combination* of elements of X is a vector of the form $\sum_{i=1}^m \alpha_i x_i$, where $x_i \in X$ and $\alpha_i \geq 0$ for all i .
- The *cone generated by* X , denoted $\text{cone}(X)$, is the set of all nonnegative combinations from X :
 - It is a convex cone containing the origin.
 - It need not be closed.
 - If X is a finite set, $\text{cone}(X)$ is closed (non-trivial to show!)

CARATHEODORY'S THEOREM



(a)



(b)

- Let X be a nonempty subset of \mathbb{R}^n .
 - (a) Every $x \neq 0$ in $\text{cone}(X)$ can be represented as a positive combination of vectors x_1, \dots, x_m from X that are linearly independent.
 - (b) Every $x \notin X$ that belongs to $\text{conv}(X)$ can be represented as a convex combination of vectors x_1, \dots, x_m from X such that $x_2 - x_1, \dots, x_m - x_1$ are linearly independent.

PROOF OF CARATHEODORY'S THEOREM

(a) Let x be a nonzero vector in $\text{cone}(X)$, and let m be the smallest integer such that x has the form $\sum_{i=1}^m \alpha_i x_i$, where $\alpha_i > 0$ and $x_i \in X$ for all $i = 1, \dots, m$. If the vectors x_i were linearly dependent, there would exist $\lambda_1, \dots, \lambda_m$, with

$$\sum_{i=1}^m \lambda_i x_i = 0$$

and at least one of the λ_i is positive. Consider

$$\sum_{i=1}^m (\alpha_i - \bar{\gamma} \lambda_i) x_i,$$

where $\bar{\gamma}$ is the largest γ such that $\alpha_i - \gamma \lambda_i \geq 0$ for all i . This combination provides a representation of x as a positive combination of fewer than m vectors of X – a contradiction. Therefore, x_1, \dots, x_m , are linearly independent.

(b) Apply part (a) to the subset of \mathfrak{R}^{n+1}

$$Y = \{(x, 1) \mid x \in X\}$$

AN APPLICATION OF CARATHEODORY

- The convex hull of a compact set is compact.

Proof: Let X be compact. We take a sequence in $\text{conv}(X)$ and show that it has a convergent subsequence whose limit is in $\text{conv}(X)$.

By Caratheodory, a sequence in $\text{conv}(X)$ can be expressed as $\left\{ \sum_{i=1}^{n+1} \alpha_i^k x_i^k \right\}$, where for all k and i , $\alpha_i^k \geq 0$, $x_i^k \in X$, and $\sum_{i=1}^{n+1} \alpha_i^k = 1$. Since the sequence

$$\left\{ (\alpha_1^k, \dots, \alpha_{n+1}^k, x_1^k, \dots, x_{n+1}^k) \right\}$$

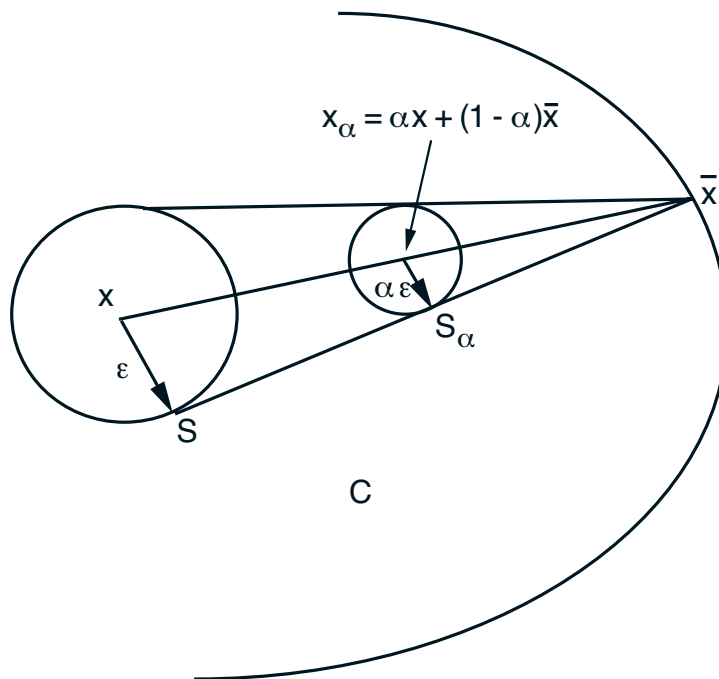
is bounded, it has a limit point

$$\left\{ (\alpha_1, \dots, \alpha_{n+1}, x_1, \dots, x_{n+1}) \right\},$$

which must satisfy $\sum_{i=1}^{n+1} \alpha_i = 1$, and $\alpha_i \geq 0$, $x_i \in X$ for all i . Thus, the vector $\sum_{i=1}^{n+1} \alpha_i x_i$, which belongs to $\text{conv}(X)$, is a limit point of the sequence $\left\{ \sum_{i=1}^{n+1} \alpha_i^k x_i^k \right\}$, showing that $\text{conv}(X)$ is compact. **Q.E.D.**

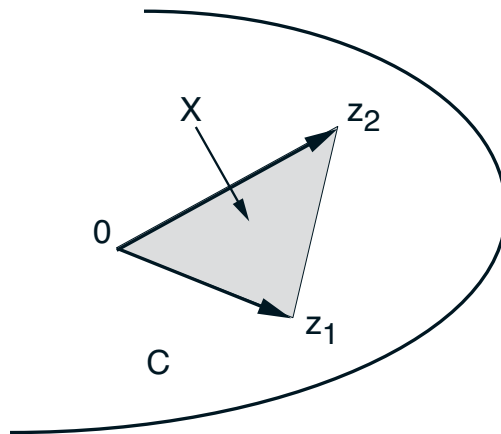
RELATIVE INTERIOR

- x is a *relative interior point* of C , if x is an interior point of C relative to $\text{aff}(C)$.
- $\text{ri}(C)$ denotes the *relative interior* of C , i.e., the set of all relative interior points of C .
- **Line Segment Principle:** If C is a convex set, $x \in \text{ri}(C)$ and $\bar{x} \in \text{cl}(C)$, then all points on the line segment connecting x and \bar{x} , except possibly \bar{x} , belong to $\text{ri}(C)$.



ADDITIONAL MAJOR RESULTS

- Let C be a nonempty convex set.
 - (a) $\text{ri}(C)$ is a nonempty convex set, and has the same affine hull as C .
 - (b) $x \in \text{ri}(C)$ if and only if every line segment in C having x as one endpoint can be prolonged beyond x without leaving C .



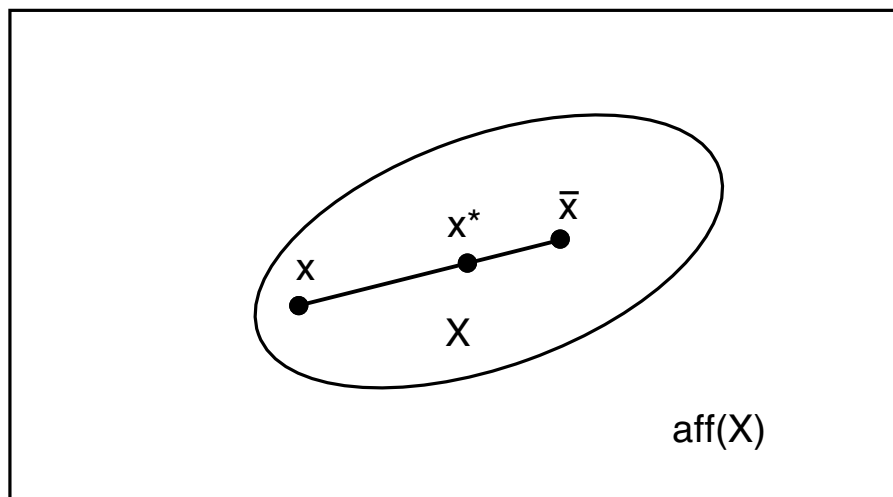
Proof: (a) Assume that $0 \in C$. We choose m linearly independent vectors $z_1, \dots, z_m \in C$, where m is the dimension of $\text{aff}(C)$, and we let

$$X = \left\{ \sum_{i=1}^m \alpha_i z_i \mid \sum_{i=1}^m \alpha_i < 1, \alpha_i > 0, i = 1, \dots, m \right\}.$$

(b) \Rightarrow is clear by the def. of rel. interior. Reverse: take any $\bar{x} \in \text{ri}(C)$; use Line Segment Principle.

OPTIMIZATION APPLICATION

- A concave function $f : \mathbb{R}^n \mapsto \mathbb{R}$ that attains its minimum over a convex set X at an $x^* \in \text{ri}(X)$ must be constant over X .



Proof: (By contradiction.) Let $x \in X$ be such that $f(x) > f(x^*)$. Prolong beyond x^* the line segment x -to- x^* to a point $\bar{x} \in X$. By concavity of f , we have for some $\alpha \in (0, 1)$

$$f(x^*) \geq \alpha f(x) + (1 - \alpha)f(\bar{x}),$$

and since $f(x) > f(x^*)$, we must have $f(x^*) > f(\bar{x})$ - a contradiction. **Q.E.D.**