LECTURE 19

LECTURE OUTLINE

- Linear and quadratic programming duality
- Conditions for existence of geometric multipliers
- Conditions for strong duality

• Primal problem: Minimize $f(x)$ subject to $x \in X$, and $g_1(x) \leq 0, \ldots, g_r(x) \leq 0$ (assuming $-\infty < f^* < \infty$). It is equivalent to $\inf_{x\in X} \sup_{\mu\geq 0} L(x,\mu)$.

• Dual problem: Maximize $q(\mu)$ subject to $\mu \geq 0$, where $q(\mu) = \inf_{x\in X} L(x,\mu)$. It is equivalent to $\sup_{\mu\geq 0} \inf_{x\in X} L(x,\mu)$.

• μ^* is a geometric multiplier if and only if $f^* = q^*$, and μ^* is an optimal solution of the dual problem.

• Question: Under what conditions $f^* = q^*$ and there exists a geometric multiplier?

LINEAR AND QUADRATIC PROGRAMMING DUALITY

- Consider an LP or positive semidefinite QP under the assumption

$$-\infty < f^* < \infty.$$

- We know from Chapter 2 that

$$-\infty < f^* < \infty \quad \Rightarrow \quad \text{there is an optimal solution } x^*.$$

- Since the constraints are linear, there exist L-multipliers corresponding to x^*, so we can use Lagrange multiplier theory.
- Since the problem is convex, the L-multipliers coincide with the G-multipliers.
- Hence there exists a G-multiplier, $f^* = q^*$ and the optimal solutions of the dual problem coincide with the Lagrange multipliers.

THE DUAL OF A LINEAR PROGRAM

- Consider the linear program

minimize $c'x$

subject to $e_i'x = d_i, \quad i = 1, \ldots, m, \qquad x \geq 0$

- Dual function

$$q(\lambda) = \inf_{x \geq 0} \left\{ \sum_{j=1}^{n} \left(c_j - \sum_{i=1}^{m} \lambda_i e_{ij} \right) x_j + \sum_{i=1}^{m} \lambda_i d_i \right\}.$$

- If $c_j - \sum_{i=1}^{m} \lambda_i e_{ij} \geq 0$ for all j, the infimum is attained for $x = 0$, and $q(\lambda) = \sum_{i=1}^{m} \lambda_i d_i$. If $c_j - \sum_{i=1}^{m} \lambda_i e_{ij} < 0$ for some j, the expression in braces can be arbitrarily small by taking x_j suff. large, so $q(\lambda) = -\infty$. Thus, the dual is

maximize $\displaystyle\sum_{i=1}^{m} \lambda_i d_i$

subject to $\displaystyle\sum_{i=1}^{m} \lambda_i e_{ij} \leq c_j, \qquad j = 1, \ldots, n.$

THE DUAL OF A QUADRATIC PROGRAM

- Consider the quadratic program

$$\text{minimize } \ \tfrac{1}{2}x'Qx + c'x$$
$$\text{subject to } \ Ax \le b,$$

where Q is a given $n \times n$ positive definite symmetric matrix, A is a given $r \times n$ matrix, and $b \in \Re^r$ and $c \in \Re^n$ are given vectors.

- Dual function:

$$q(\mu) = \inf_{x \in \Re^n} \left\{ \tfrac{1}{2}x'Qx + c'x + \mu'(Ax - b) \right\}.$$

The infimum is attained for $x = -Q^{-1}(c + A'\mu)$, and, after substitution and calculation,

$$q(\mu) = -\tfrac{1}{2}\mu' AQ^{-1}A'\mu - \mu'(b + AQ^{-1}c) - \tfrac{1}{2}c'Q^{-1}c.$$

- The dual problem, after a sign change, is

$$\text{minimize } \ \tfrac{1}{2}\mu'P\mu + t'\mu$$
$$\text{subject to } \ \mu \ge 0,$$

where $P = AQ^{-1}A'$ and $t = b + AQ^{-1}c$.

RECALL NONLINEAR FARKAS' LEMMA

Let $C \subset \Re^n$ be convex, and $f : C \mapsto \Re$ and $g_j : C \mapsto \Re$, $j = 1, \ldots, r$, be convex functions. Assume that

$$f(x) \geq 0, \qquad \forall\, x \in F = \{x \in C \mid g(x) \leq 0\},$$

and one of the following two conditions holds:

(1) 0 is in the relative interior of the set $D = \{u \mid g(x) \leq u \text{ for some } x \in C\}$.

(2) The functions g_j, $j = 1, \ldots, r$, are affine, and F contains a relative interior point of C.

Then, there exist scalars $\mu_j^* \geq 0$, $j = 1, \ldots, r$, s. t.

$$f(x) + \sum_{j=1}^{r} \mu_j^* g_j(x) \geq 0, \qquad \forall\, x \in C.$$

APPLICATION TO CONVEX PROGRAMMING

Consider the problem

minimize $f(x)$

subject to $x \in C, \quad g_j(x) \leq 0, \ j = 1, \ldots, r,$

where C, $f : C \mapsto \Re$, and $g_j : C \mapsto \Re$ are convex. Assume that the optimal value f^* is finite.

- Replace $f(x)$ by $f(x) - f^*$ and assume that the conditions of Farkas' Lemma are satisfied. Then there exist $\mu_j^* \geq 0$ such that

$$f^* \leq f(x) + \sum_{j=1}^{r} \mu_j^* g_j(x), \qquad \forall \, x \in C.$$

Since $F \subset C$ and $\mu_j^* g_j(x) \leq 0$ for all $x \in F$,

$$f^* \leq \inf_{x \in F} \left\{ f(x) + \sum_{j=1}^{r} \mu_j^* g_j(x) \right\} \leq \inf_{x \in F} f(x) = f^*.$$

Thus equality holds throughout, we have

$$f^* = \inf_{x \in C} \left\{ f(x) + \mu^{*\prime} g(x) \right\},$$

and μ^* is a geometric multiplier.

STRONG DUALITY THEOREM I

Assumption : (Convexity and Linear Constraints) f^* is finite, and the following hold:

(1) $X = P \cap C$, where P is polyhedral and C is convex.

(2) The cost function f is convex over C and the functions g_j are affine.

(3) There exists a feasible solution of the problem that belongs to the relative interior of C.

Proposition : Under the above assumption, there exists at least one geometric multiplier.

Proof: If $P = \Re^n$ the result holds by Farkas. If $P \neq \Re^n$, express P as

$$P = \{x \mid a_j' x - b_j \leq 0,\ j = r+1, \ldots, p\}.$$

Apply Farkas to the extended representation, with

$$F = \{x \in C \mid a_j' x - b_j \leq 0,\ j = 1, \ldots, p\}.$$

Assert the existence of geometric multipliers in the extended representation, and pass back to the original representation. **Q.E.D.**

STRONG DUALITY THEOREM II

Assumption : (Linear and Nonlinear Constraints) f^* is finite, and the following hold:

(1) $X = P \cap C$, with P: polyhedral, C: convex.

(2) The functions f and g_j, $j = 1, \ldots, \overline{r}$, are convex over C, and the functions g_j, $j = \overline{r} + 1, \ldots, r$ are affine.

(3) There exists a feasible vector $\bar{x}$ such that $g_j(\bar{x}) < 0$ for all $j = 1, \ldots, \overline{r}$.

(4) There exists a vector that satisfies the linear constraints [but not necessarily the constraints $g_j(x) \leq 0$, $j = 1, \ldots, \overline{r}$] and belongs to the relative interior of C.

Proposition : Under the above assumption, there exists at least one geometric multiplier.

Proof: If $P = \Re^n$ and there are no linear constraints (the Slater condition), apply Farkas. Otherwise, lump the linear constraints within X, assert the existence of geometric multipliers for the nonlinear constraints, then use the preceding duality result for linear constraints. **Q.E.D.**

THE PRIMAL FUNCTION

- Minimax theory centered around the function

$$p(u) = \inf_{x \in X} \sup_{\mu \geq 0} \big\{ L(x, \mu) - \mu' u \big\}$$

- Properties of p around $u = 0$ are critical in analyzing the presence of a duality gap and the existence of primal and dual optimal solutions.

- p is known as the *primal function* of the constrained optimization problem.

- We have

$$\sup_{\mu \geq 0} \big\{ L(x, \mu) - \mu' u \big\}$$
$$= \sup_{\mu \geq 0} \big\{ f(x) + \mu' \big(g(x) - u \big) \big\}$$
$$= \begin{cases} f(x) & \text{if } g(x) \leq u, \\ \infty & \text{otherwise,} \end{cases}$$

- So

$$p(u) = \inf_{\substack{x \in X \\ g(x) \leq u}} f(x)$$

and $p(u)$ can be viewed as a *perturbed optimal value* [note that $p(0) = f^*$].

CONDITIONS FOR NO DUALITY GAP

- Apply the minimax theory specialized to $L(x, \mu)$.
- Assume that $f^* < \infty$, and that X is convex, and $L(\cdot, \mu)$ is convex over X for each $\mu \geq 0$. Then:
  - p is convex.
  - There is no duality gap if and only if p is lower semicontinuous at $u = 0$.
- Conditions that guarantee lower semicontinuity at $u = 0$, correspond to those for preservation of closure under partial minimization, e.g.:
  - $f^* < \infty$, X is convex and compact, and for each $\mu \geq 0$, the function $L(\cdot, \mu)$, restricted to have domain X, is closed and convex.
  - Extensions involving directions of recession of X, f, and g_j, and guarantee that the minimization in $p(u) = \inf_{\substack{x \in X \\ g(x) \leq u}} f(x)$ is (effectively) over a compact set.
- Under the above conditions, there is no duality gap, and the primal problem has a nonempty and compact optimal solution set. Furthermore, the primal function p is closed, proper, and convex.