LECTURE 2

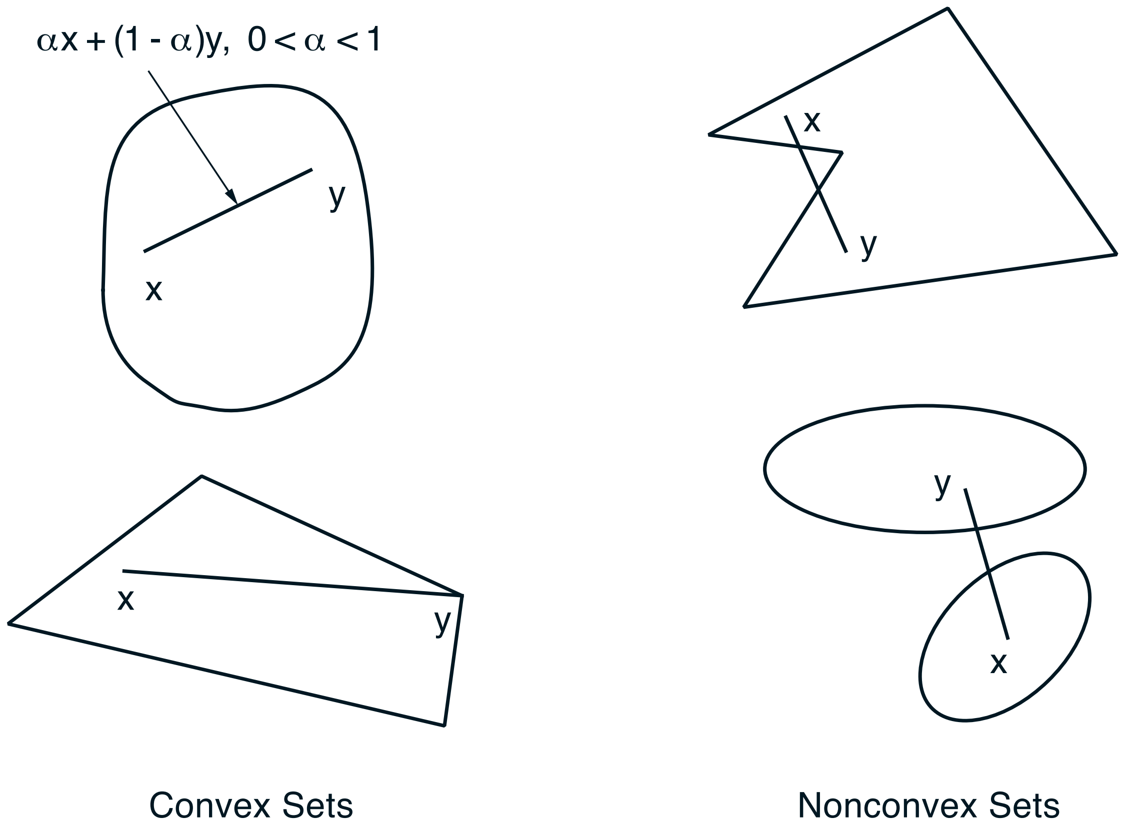
LECTURE OUTLINE

- Convex sets and Functions
- Epigraphs
- Closed convex functions
- Recognizing convex functions

SOME MATH CONVENTIONS

- All of our work is done in $\Re^n$: space of n-tuples $x = (x_1, \ldots, x_n)$
- All vectors are assumed column vectors
- "$'$" denotes transpose, so we use x' to denote a row vector
- $x'y$ is the inner product $\sum_{i=1}^{n} x_i y_i$ of vectors x and y
- $\|x\| = \sqrt{x'x}$ is the (Euclidean) norm of x. We use this norm almost exclusively
- See Section 1.1 of the book for an overview of the linear algebra and real analysis background that we will use

CONVEX SETS



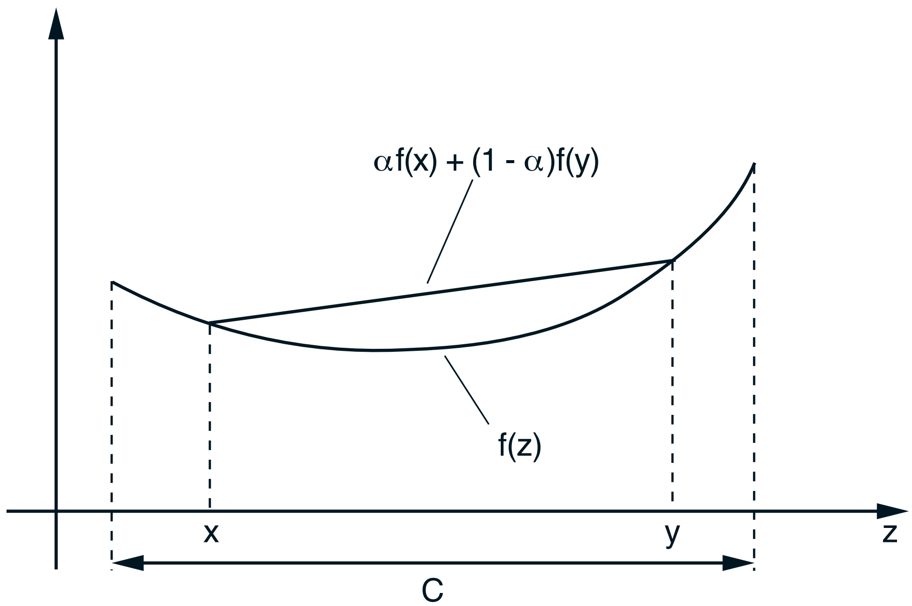
- A subset C of $\Re^n$ is called $convex$ if

$$\alpha x + (1 - \alpha) y \in C, \qquad \forall\, x, y \in C,\ \forall\, \alpha \in [0, 1]$$

- Operations that preserve convexity
  - Intersection, scalar multiplication, vector sum, closure, interior, linear transformations

- Cones: Sets C such that $\lambda x \in C$ for all $\lambda > 0$ and $x \in C$ (not always convex or closed)

CONVEX FUNCTIONS



- Let C be a convex subset of $\Re^n$. A function $f : C \mapsto \Re$ is called *convex* if

$$f\big(\alpha x + (1-\alpha)y\big) \leq \alpha f(x) + (1-\alpha)f(y), \qquad \forall\, x, y \in C$$

- If f is a convex function, then all its level sets $\{x \in C \mid f(x) \leq a\}$ and $\{x \in C \mid f(x) < a\}$, where a is a scalar, are convex.

EXTENDED REAL-VALUED FUNCTIONS

- The *epigraph* of a function $f : X \mapsto [-\infty, \infty]$ is the subset of $\Re^{n+1}$ given by

$$\mathrm{epi}(f) = \big\{(x, w) \mid x \in X,\ w \in \Re,\ f(x) \le w\big\}$$

- The *effective domain* of f is the set

$$\mathrm{dom}(f) = \big\{x \in X \mid f(x) < \infty\big\}.$$

- We say that f is *proper* if $f(x) < \infty$ for at least one $x \in X$ and $f(x) > -\infty$ for all $x \in X$, and we will call f *improper* if it is not proper. Thus f is proper if and only if its epigraph is nonempty and does not contain a “vertical line.”

- An extended real-valued function $f : X \mapsto [-\infty, \infty]$ is called *lower semicontinuous* at a vector $x \in X$ if $f(x) \le \liminf_{k\to\infty} f(x_k)$ for every sequence $\{x_k\} \subset X$ with $x_k \to x$.

- We say that f is *closed* if $\mathrm{epi}(f)$ is a closed set.

CLOSEDNESS AND SEMICONTINUITY

• *Proposition*: For a function $f : \Re^n \mapsto [-\infty, \infty]$, the following are equivalent:

(i) The level set $\{x \mid f(x) \leq a\}$ is closed for every scalar a.

(ii) f is lower semicontinuous over $\Re^n$.

(iii) $\mathrm{epi}(f)$ is closed.

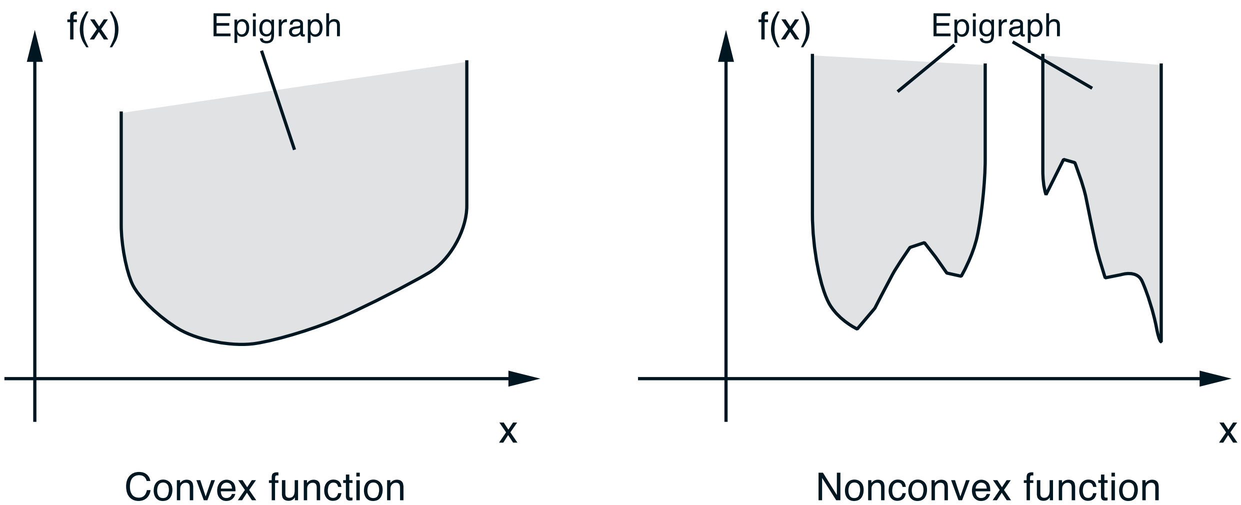
• Note that:

– If f is lower semicontinuous over $\mathrm{dom}(f)$, it is not necessarily closed

– If f is closed, $\mathrm{dom}(f)$ is not necessarily closed

• *Proposition*: Let $f : X \mapsto [-\infty, \infty]$ be a function. If $\mathrm{dom}(f)$ is closed and f is lower semicontinuous over $\mathrm{dom}(f)$, then f is closed.

EXTENDED REAL-VALUED CONVEX FUNCTIONS



- Let C be a convex subset of $\Re^n$. An extended real-valued function $f : C \mapsto [-\infty, \infty]$ is called *convex* if $\text{epi}(f)$ is a convex subset of $\Re^{n+1}$.
- If f is proper, this definition is equivalent to

$$f\big(\alpha x+(1-\alpha)y\big) \leq \alpha f(x)+(1-\alpha)f(y), \qquad \forall\, x, y \in C$$

- An improper closed convex function is very peculiar: it takes an infinite value (∞ or $-\infty$) at every point.

RECOGNIZING CONVEX FUNCTIONS

• Some important classes of elementary convex functions: Affine functions, positive semidefinite quadratic functions, Euclidean norm, etc

• *Proposition*: Let $f_i : \Re^n \mapsto (-\infty, \infty]$, $i \in I$, be given functions (I is an arbitrary index set).
(a) The function $g : \Re^n \mapsto (-\infty, \infty]$ given by

$$g(x) = \lambda_1 f_1(x) + \cdots + \lambda_m f_m(x), \qquad \lambda_i > 0$$

is convex (or closed) if $f_1, \ldots, f_m$ are convex (respectively, closed).
(b) The function $g : \Re^n \mapsto (-\infty, \infty]$ given by

$$g(x) = f(Ax)$$

where A is an $m \times n$ matrix is convex (or closed) if f is convex (respectively, closed).
(c) The function $g : \Re^n \mapsto (-\infty, \infty]$ given by

$$g(x) = \sup_{i \in I} f_i(x)$$

is convex (or closed) if the f_i are convex (respectively, closed).